

S5: Diagonalization

Dfn: A linear transformation $T: V \rightarrow V$ is called diagonalizable if there exists a basis $\beta = \{v_i\}$

such that $T(v_i) = \lambda_i v_i$ for some scalars $\lambda_i \in \mathbb{R}$.

$(\Rightarrow)$ $[T]_\beta$ is a diagonal matrix.

• An $n \times n$ matrix A is diagonalizable

$(\Rightarrow) L_A: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is diagonalizable.

Ex: $T: P_2 \rightarrow P_2$ defined by

$$(Tf)(x) = f(x) + (x+1)f'(x).$$

Consider the basis $\beta = \left\{ \underset{v_1}{1}, \underset{v_2}{x+1}, \underset{v_3}{(x+1)^2} \right\}$.

$$T(1) = 1 + 0 = 1, \quad \lambda_1 = 1.$$

$$T(x+1) = x+1 + (x+1) \cdot 1 = 2(x+1), \quad \lambda_2 = 2.$$

$$T((x+1)^2) = (x+1)^2 + (x+1) \cdot (2(x+1)) = 3(x+1)^2, \quad \lambda_3 = 3.$$

In other words:

$$[T]_\beta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Let $T: V \rightarrow V$ be a general linear transformation.

Dfn: A vector v is an eigenvector for T if
there exists $\lambda \in \mathbb{R}$ s.t.

$$T(v) = \lambda v.$$

The scalar λ is called the corresponding eigenvalue.

Note: "Eigen" is German for "own" or "characteristic".

The same terminology is used for column vectors: an
eigenvector for an $n \times n$ matrix A is $\vec{v} \in \mathbb{R}^n$ s.t.

$$A \cdot \vec{v} = \lambda \vec{v}.$$

Theorem: $T: V \rightarrow V$ is diagonalizable $\Leftrightarrow$ there
exists a basis $\beta = \{v_1, \dots, v_n\}$ consisting of
eigenvectors for T .

To "diagonalize" is to find such a basis.

Ex: $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$. $\{\vec{v}_1, \vec{v}_2\} = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$.

$$A\vec{v}_1 = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \quad \lambda_1 = -2.$$

$$A\vec{v}_2 = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 15 \\ 20 \end{bmatrix} = 5 \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \quad \lambda_2 = 5.$$

So A is diagonalizable.

$$[L_A]_{\beta} = \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix} = D$$

Note:

$$\begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix} = Q^{-1} A Q$$

where $Q = \begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix} = [\vec{v}_1 | \vec{v}_2]$

Thm: Suppose A is diagonalizable with eigenbasis

$\{\vec{v}_1, \dots, \vec{v}_n\}$ and corresponding eigenvalues $\{\lambda_1, \dots, \lambda_n\}$.

Then

$$\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = Q^{-1} A Q, \quad Q = [\vec{v}_1 | \dots | \vec{v}_n]$$

Pf: Q is called the "change of basis" matrix.

Note

$$Q = \begin{bmatrix} \mathbb{1} \end{bmatrix}_{\beta_{std} \beta}^{\beta}, \quad \beta_{std} = \{e_1, \dots, e_n\}.$$

$$A = \begin{bmatrix} L_A \end{bmatrix}_{\beta_{std} \beta_{std}}^{\beta_{std} \beta_{std}}.$$

So:

$$Q^{-1} A Q = \begin{bmatrix} \mathbb{1} \end{bmatrix}_{\beta}^{\beta_{std}} \cdot \begin{bmatrix} L_A \end{bmatrix}_{\beta_{std} \beta_{std}}^{\beta_{std} \beta_{std}} \cdot \begin{bmatrix} \mathbb{1} \end{bmatrix}_{\beta_{std}}^{\beta}$$

$$= \begin{bmatrix} \mathbb{1} \circ L_A \circ \mathbb{1} \end{bmatrix}_{\beta}^{\beta}$$

$$= \begin{bmatrix} L_A \end{bmatrix}_{\beta} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$



Ex: Consider the vector space $V = C^\infty(\mathbb{R})$ of all smooth fns $f: \mathbb{R} \rightarrow \mathbb{R}$. Define

$$T: C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R}).$$

by $T(f) = f'$.

Then, f is an eigenvalue if $f' = \lambda f$.

The solution to this ODE is

$$f = c e^{\lambda t} \quad //$$

A method to finding eigenvalues.

Dfn: $A \in M_{n \times n}$. The characteristic polynomial of A is

$$\chi_A(t) = \det(A - t \mathbb{I}_{n \times n}).$$

"chi"

Theorem: $\chi_A(t) \in \mathbb{P}_n$. Also, $\lambda \in \mathbb{R}$ is an eigenvalue of $A \Leftrightarrow \chi_A(\lambda) = 0$.

Pf : Suppose $\lambda \in \mathbb{R}$ is eigenvalue :

$$Av = \lambda v \text{ for some nonzero } v \in \mathbb{R}^n$$

Then

$$Av - \lambda v = 0$$

$$\text{or } (A - \lambda I_{n \times n})v = 0$$

$\Rightarrow A - \lambda I_{n \times n}$ is not invertible since $v \in \ker(A - \lambda I)$

and $v \neq 0$. Thus $\det(A - \lambda I_{n \times n}) = 0$. $\square$

Ex : $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$. Then

$$\begin{aligned} \chi_A(t) &= \det \begin{bmatrix} 1-t & 1 \\ 4 & 1-t \end{bmatrix} = t^2 - 2t - 3 \\ &= (t-3)(t+1) \end{aligned}$$

So

$\lambda = 3, -1$ are eigenvalues.

To find corresponding eigenvector we solve

$$(A - \lambda I)v = 0$$

$$\underline{\lambda = 3} : \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow x_2 = 2x_1$$

An eigenvector is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

$$\underline{\lambda = -1} : \begin{bmatrix} +2 & 1 \\ 4 & +2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}.$$

So eigenvector is $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Eigenbasis is thus $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$.