

Cayley-Hamilton.

Let $T: V \rightarrow V$
 $\dim V = n$

Recall A T -invariant subspace

$$W \subset V$$

is one such that $T(W) \subset W$.

Last time: two main results

① If $W \subset V$ is T invariant and

$$T|_W: W \rightarrow W$$

is the restriction. Then

$\chi_{T|_W}(t)$ divides $\chi_T(t)$

~~① If $W \subset V$~~

② If $x \in V$ is non-zero then a basis
for the T -invariant subspace

$$W = \text{span} \{ x, Tx, T^2x, \dots \}$$

is

$$\{ x, Tx, T^2x, \dots, T^{k-1}x \}$$

where $k = \dim W$.

Theorem: If $v \in V$ is nonzero and

$$(*) \quad a_0 v + a_1 T(v) + \dots + a_{k-1} T^{k-1}(v) + T^k(v) = 0$$

then

$$\chi_{T|W}(t) = (-1)^k (a_0 + a_1 t + \dots + a_{k-1} t^{k-1} + t^k)$$

Pf: Let $k = \dim W$ and

$$\beta = \{v, T(v), \dots, T^{k-1}(v)\}$$

be the ordered basis for W that

we used last time. Note that the

formula (*) is equivalent to

$$T^k(v) = -a_0 v - a_1 T(v) - \dots - a_{k-1} T^{k-1}(v).$$

Next, we compute $[T]_\beta$.

~~$T(v) \mapsto T^2(v)$~~

$$v \mapsto T(v)$$

$$T(v) \mapsto T^2(v) \dots$$

and of the last column:

$$T^{k-1}(v) \mapsto T^k(v) = -a_0 v - \dots - a_{k-1} T^{k-1}(v)$$

So

$$[T]_{\beta} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & 0 & \dots & 0 & -a_2 \\ 0 & 0 & 1 & \dots & 0 & -a_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -a_{k-1} \end{bmatrix}$$

We want to compute

$$\det([T]_{\beta} - t \mathbb{I}) \quad \text{which we do by expanding along last column.}$$

$$(-1)^{k-1} (-a_0) \det \begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \quad R1$$

$$(-1)^{k-1} (-1) (-a_1) \det \begin{bmatrix} -t & & * \\ & 1 & \\ 0 & & 1 \end{bmatrix} \quad R2$$

$$(-1)^{k-1} (-1)^2 (-a_2) \det \begin{bmatrix} -t & & * \\ & -t & \\ & & 1 \\ 0 & & & 1 \end{bmatrix} \quad R3$$

$$(-1)^{k-1} (-1)^{k-1} (-a_{k-1} - t) \det \begin{bmatrix} -t & & \\ & \ddots & \\ & & -t \end{bmatrix}$$

$$= (-1)^{k-1} \left(a_0 + a_1 t + \dots + a_{k-1} t^{k-1} + \cancel{t^k} t^k \right)$$



Theorem: Let A be any $n \times n$ matrix and let

$$\chi_A(t) = (-1)^k \left(t^k + a_{k-1} t^{k-1} + \dots + a_1 t + a_0 \right)$$

be its characteristic polynomial. Then

$$\chi_A(A) = (-1)^k \left(A^k + a_{k-1} A^{k-1} + \dots + a_1 A + a_0 \mathbb{I} \right)$$

$$= \underline{\underline{0}}$$

the zero matrix.

~~QED~~ or: If $T: V \rightarrow V$ is any linear transformation and $\chi_T(t)$ is char polynomial. Then $\chi_T(T) = 0$.

Pf: Spc $v \neq 0$, let

$$W = \text{span}\{v, Tv, T^2v, \dots\}.$$

We know $\{v, Tv, \dots, T^{k-1}v\}$ is a basis for W . Thus we can find scalars s.t.

$$a_0 v + a_1 Tv + \dots + a_{k-1} T^{k-1}v + T^k v = 0.$$

The last theorem then implies that

$$\chi_{T|_W}(t) = (-1)^k (a_0 + a_1 t + \dots + a_{k-1} t^{k-1} + t^k).$$

So we have shown that

$$\chi_{T|_W}(T)(v) = 0.$$

for any nonzero v .

We also know that

~~$\chi_{T|w}(t)$~~ divides $\chi_T(t)$.

So

$$\chi_T(t) = \chi_{T|w}(t) \cdot g(t) = g(t) \chi_{T|w}(t)$$

for some polynomial g . Thus

~~$$\chi_T(T)(v) = \chi_{T|w}(T)(v) g(T)(v)$$~~

~~$\chi_T(T)(v)$~~

$$\begin{aligned} \chi_T(T)(v) &= g(T) \left(\chi_{T|w}(T)(v) \right) \\ &= g(T)(0) = 0. \end{aligned}$$

$\square$