

MA 442 - Quiz

March 25

Name: _____ BUID: _____

Question 1. Let $a \in \mathbb{R}$ be a real scalar. Find the rank of the matrix $A = \begin{bmatrix} 1 & a & a^2 \\ a & a^2 & 1 \end{bmatrix}$. (Hint: The rank of the matrix depends on the value of a , you must determine how the rank depends on a .)

Solution. Suppose $a = 0$. Then the matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ which is clearly rank 2. If $a \neq 0$ then we can perform row operations of type 2 and type 3 to obtain

$$\begin{bmatrix} 1 & a & a^2 \\ a & a^2 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & a & a^2 \\ 0 & a^2 & 1 - a^3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & a & a^2 \\ 0 & 0 & 1 - a^3 \end{bmatrix}. \quad (1)$$

Recall, we are assuming that $a \neq 0$, thus this matrix has rank 2 for all values of a except $1 - a^3 = 0$ or $a = 1$. Thus $\text{rank}(A) = 2$ if $a \neq 1$ and $\text{rank}(A) = 1$ if $a = 1$.

Question 2. The following matrix

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad (2)$$

is invertible. Find B^{-1} .

Solution.

$$\begin{aligned} & \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right] \\ & \xrightarrow{R_2 \rightarrow R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right] \\ & \xrightarrow{R_3 \rightarrow R_3 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & -1 & 1 \end{array} \right] \\ & \xrightarrow{R_3 \rightarrow \frac{1}{2}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array} \right] \\ & \xrightarrow{R_1 \rightarrow R_1 - R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array} \right] \\ & \xrightarrow{R_2 \rightarrow R_2 + R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array} \right] \end{aligned}$$

Thus,

$$B^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}.$$