HOMEWORK 2
DUE SEPTEMBER 22

There are three problems to turn in.

Note: The notation below (specifically in problem 2) differs slightly from the notation used in class. If $F\colon U \to \mathbf{R}^m$ is a differentiable function defined on an open set $U \subset \mathbf{R}^n$ then for $p \in U$ we denote by

$$D_p F\colon \mathbf{R}^n \to \mathbf{R}^m \tag{1}$$

the total derivative of F at p. In class we used the notation $DF(p)$.

(1) Let $M_n(\mathbf{R})$ denote the vector space of real $n \times n$ matrices.
   (a) Fix a matrix $C \in M_n(\mathbf{R})$ and define the map $l_C\colon M_n(\mathbf{R}) \to M_n(\mathbf{R})$ by the rule $l_C(A) = CA$. Show that l_C is differentiable and find its derivative.
   (b) Let $\tau\colon M_n(\mathbf{R}) \to M_n(\mathbf{R})$ be the transpose map $\tau(A) = A^t$. Show that τ is differentiable and find its derivative.
   (c) Let $f, g\colon M_n(\mathbf{R}) \to M_n(\mathbf{R})$ be differentiable maps. Show that the map $h\colon M_n(\mathbf{R}) \to M_n(\mathbf{R})$ defined by $h(A) = f(A)g(A)$ is differentiable and express its derivative in terms of the derivatives of f, g.
   (d) Let $f\colon M_n(\mathbf{R}) \to M_n(\mathbf{R})$ be the map $f(A) = A^t A$. Show that f is differentiable and find its derivative.

(2) The determinant function

$$\det\colon M_n(\mathbf{R}) \to \mathbf{R}$$

is a polynomial in n variables; as such it is a smooth function. Complete the steps below to express the derivative of det in terms of familiar objects.
   (a) Let $\mathbb{1} \in M_n(\mathbf{R})$ be the identity matrix. For a matrix B compute $D_{\mathbb{1}}(\det)(B)$ in terms of a simple invariant of $n \times n$ matrices.
   (b) Using (a), find an expression for $D_A(\det)(B)$ where $A \in GL_n(\mathbf{R}) \subset M_n(\mathbf{R})$ is an invertible $n \times n$ matrix.
   (c) Let $\operatorname{cof}(A)$ denote the cofactor of a square matrix. Using that $GL_n(\mathbf{R}) \subset M_n(\mathbf{R})$ is an open dense subset (you may use this without proof) find a formula for $D_A(\det)(B)$ for arbitrary $A, B \in M_n(\mathbf{R})$ in terms of $\operatorname{cof}(A)$. (Hint: when $A \in GL_n(\mathbf{R})$ one has $\operatorname{cof}(A) = (\det A)A^{-1}$).

(3) Let M be any topological space and let $C^0(M)$ denote the algebra of continuous functions on M. Given a continuous map between spaces $F\colon M \to N$ define $F^*\colon C^0(N) \to C^0(M)$ by $F^*(f) = f \circ F$. We say that $F^*(f)$ is the *pullback* (or restriction) of f along F.
   (a) Show that F^* is an algebra homomorphism.
   (b) Suppose now that M, N are smooth manifolds. Show that $F\colon M \to N$ is smooth if and only if
   $$F^*\left(C^\infty(N)\right) \subset C^\infty(M).$$
   (c) Show that $F^*\colon C^\infty(N) \to C^\infty(M)$ is an isomorphism if F is a diffeomorphism.