SOLUTIONS TO HOMEWORK 2

Problem 1.

(a) The map l_C is linear hence differentiable (in fact, smooth). The derivative is

$$Dl_C|_A(B) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} l_C(A + tB) = CB. \tag{1}$$

(b) The map τ is linear and hence differentiable. The derivative is

$$D\tau|_A(B) = B^t. \tag{2}$$

(c) We show that h is differentiable with derivative

$$Dh|_A(B) = (Df|_A(B))g(A) + f(A)(Dg|_A(B)). \tag{3}$$

Denote this linear map by T. It suffices to show that

$$\frac{f(A+H)g(A+H) - f(A)g(A) - T(H)}{\|H\|} \to 0 \tag{4}$$

as $H \to 0$. Notice that

$$\begin{aligned} & f(A+H)g(A+H) - f(A)g(A) - TH = \\ & (f(A+H) - f(A) - Df|_A(H))\, g(A+H) + f(A)\,(g(A+H) - g(A) - Dg|_A(H)) \\ & \qquad + Df|_A(H)\,(g(A+H) - g(A)). \end{aligned} \tag{5}$$

Dividing by $\|H\|$ we see that the second line approaches zero as $H \to 0$ by differentiability of f, g respectively. It suffices to see that

$$\frac{Df|_A(H)\,(g(A+H) - g(A))}{\|H\|} \to 0 \tag{6}$$

as $H \to 0$. For this we choose a constant C such that $\|Df|_A(H)\| \leq C\|H\|$ for all $n \times n$ matrices H, which exists since $Df|_A$ is a linear map. Then

$$\frac{\|Df|_A(H)\,(g(A+H) - g(A))\,\|}{\|H\|} \leq C\|g(A+H) - g(A)\|. \tag{7}$$

This approaches zero as $H \to 0$ by continuity of g.

(d) By (b),(c) the map f is differentiable with derivative

$$Df|_A(B) = B^t A + A^t B. \tag{8}$$

Problem 2.

(a) The stated derivative is

$$D \det|_{\mathbb{1}}(B) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} \det(\mathbb{1} + tB). \tag{9}$$

For $t \neq 0$ we have

$$\det(\mathbb{1} + tB) = t^n \det(t^{-1}\mathbb{1} + B) = t^n p_{-B}(t^{-1}) \tag{10}$$

where p_{-B} is the characteristic polynomial associated to the matrix $-B$. In derivative above it is immediate that only the coefficient of s^{n-1} in $p_{-B}(s)$ contributes; it is a standard fact that this coefficient is $-\operatorname{tr}(-B) = \operatorname{tr}(B)$. Thus

$$D \det|_{\mathbb{1}}(B) = \operatorname{tr}(B). \tag{11}$$

(b) Suppose A is invertible. Then

$$\det(A + tB) = \det\left[A(\mathbb{1} + tA^{-1}B)\right]. \tag{12}$$

But since $\det(XY) = \det X \det Y$ we see that

$$\det(A + tB) = (\det A)\det(\mathbb{1} + tA^{-1}B). \tag{13}$$

It follows from part (a) that

$$D \det|_A(B) = (\det A) D \det|_{\mathbb{1}}(A^{-1}B) = (\det A)\operatorname{tr}(A^{-1}B). \tag{14}$$

(c) The cofactor matrix of an invertible matrix A satisfies

$$\operatorname{cof}(A) = (\det A)A^{-1}. \tag{15}$$

So from part (b) we see that if A is invertible then

$$D \det|_A(B) = \operatorname{tr}(\operatorname{cof}(A)B). \tag{16}$$

As invertible matrices are dense in all matrices, the same formula gives the expression for the derivative of det at any matrix.

Problem 3.

(a) Let f, g be continuous functions on N. Their product fg is also continuous. Moreover if $x \in M$ we have

$$F^*(fg)(x) = (fg) \circ F(x) = f(F(x))g(F(x)) = (F^*f)(x)(F^*g)(x). \tag{17}$$

This shows taht F^* is an algebra homomorphism.

(b) Suppose $f \in C^\infty(N)$. Then by the chain rule we know that $F^*f = f \circ F$ is a smooth function on M.

Conversely suppose that for every $f \in C^\infty(N)$ we have $F^*f \in C^\infty(M)$. We want to show that F is smooth. Let M, N have dimensions m, n respectively. Take a chart (V, ψ) for N and let $\widetilde{U} = F^{-1}(V) \subset M$; since F is continuous this is an open subset. Since M is smooth there exists a possibly smaller open subset $U \subset \widetilde{U}$ equipped with a coordinate chart $\phi \colon U \to \mathbf{R}^m$. By construction $F(U) \subset V$. Consider the composition $\psi \circ F \circ \phi^{-1}$:

$$\phi(U) \xrightarrow{\phi^{-1}} U \xrightarrow{F} V \xrightarrow{\psi} \psi(V). \tag{18}$$

Since ψ is smooth we know that $F^*\psi = \psi \circ F \colon U \to \psi(V)$ is also smooth by assumption. Since ϕ^{-1} is smooth the entire composition above is smooth.

[(c)] Suppose F is a diffeomorphism. The inverse to F^* is $(F^{-1})^*$. In other words $(F^*)^{-1} = (F^{-1})^*$.

As mentioned in class, to prove the converse we must assume that F is a homeomorphism so that its has a continuous inverse F^{-1}. Since F^* maps $C^\infty(N) \subset C^0(N)$ isomorphically to $C^\infty(M) \subset C^0(M)$ we see that by part (b) that F is smooth. Conversely $(F^{-1})^*$ maps $C^\infty(M) \subset C^0(M)$ isomorphically to $C^\infty(N) \subset C^0(N)$ we see that F^{-1} is smooth.