HOMEWORK 3
DUE SEPTEMBER 29

There are three problems to turn in.

(1) Let M be a connected smooth manifold, N be any smooth manifold, and $F\colon M \to N$ a smooth map. Show that the differential $\mathrm{d}F_p$ is the zero map for each $p \in M$ if and only if F is a constant map. (Recall that we say F is constant if there exists a $q \in N$ such that $F(x) = q$ for all $x \in M$.)

(2) $\mathbf{C}^n = \mathbf{C} \times \cdots \times \mathbf{C}$ is the space of n-tuples of complex numbers; it is naturally a complex vector space. Using the identification $\mathbf{C} = \mathbf{R}^2$:

$$z = x + \mathrm{i}y \leftrightarrow (x, y)$$

we see that $\mathbf{C}^n$ is naturally a smooth manifold diffeomorphic to $\mathbf{R}^{2n}$.

Let $\mathbf{CP}^1$ be the set of complex one-dimensional linear subspaces of $\mathbf{C}^2$.

(a) Define an equivalence relation on $\mathbf{C}^2 \setminus \{0\}$ by

$$(z, w) \sim (z', w') \iff (z', w') = (\lambda z, \lambda w), \quad \lambda \in \mathbf{C}.$$

Show that there is a bijection

$$\mathbf{CP}^1 \cong \left(\mathbf{C}^2 \setminus \{0\}\right) / \sim .$$

(b) We endow $\mathbf{CP}^1$ with the *quotient topology*, described as follows. Let $\pi\colon \mathbf{C}^2 \setminus \{0\} \to \mathbf{CP}^1$ be the map which sends a nonzero vector v to its equivalence class $[v]$ as in part (a). We say $U \subset \mathbf{CP}^1$ is open if and only if

$$\pi^{-1}(U) \subset \mathbf{C}^2 \setminus \{0\}$$

is open.[1] Let $V_z \subset \mathbf{C}^2 \setminus \{0\}$ be the set where $z \neq 0$ and let $V_w \subset \mathbf{C}^2 \setminus \{0\}$ be the set where $w \neq 0$. Show that $U_z = \pi(V_z)$ and $U_w = \pi(V_w)$ are open subsets of $\mathbf{CP}^1$.

(c) Define $\phi_z\colon U_z \to \mathbf{C}$ by the formula

$$\phi_z([z, w]) = \frac{w}{z}$$

and $\phi_w\colon U_w \to \mathbf{C}$ by

$$\phi_w([z, w]) = \frac{z}{w}.$$

[1] This topology is Hausdorff and second countable. You do not need to prove this.

Show that ϕ_z, ϕ_w endow M with the structure of a topological manifold of dimension two.

(d) Show that the atlas $\{(U_z, \phi_z), (U_w, \phi_w)\}$ is smooth, hence endowing $\mathbf{CP}^1$ with the structure of a smooth manifold. This smooth manifold is called "complex projective space".

(3) You may treat all of the assertions in problem (2) as proven.

(a) Show that $\pi\colon \mathbf{C}^2 \setminus \{0\} \to \mathbf{CP}^1$ is a smooth submersion.

(b) Using (a), argue that there exists a smooth map $p\colon S^3 \to \mathbf{CP}^1$ with the property that $p^{-1}([z,w]) \cong S^1$ for every $[z,w] \in \mathbf{CP}^1$.

(c) Consider the subset $p^{-1}([0,1]) \subset S^3$ which by (b) is homeomorphic to S^1. Consider the map

$$F\colon S^3 \setminus p^{-1}([0,1]) \to S^2$$

defined by

$$F(z,w) = \frac{1}{1+|z/w|^2}\left(z/w + \overline{z}/\overline{w}, -\mathrm{i}(z/w - \overline{z}/\overline{w}), |z/w|^2 - 1\right).$$

Show that F is well-defined and extends to a map $\widetilde{F}\colon S^3 \to S^2$.

(d) (BONUS: Does not need to be turned in to receive full credit) Construct a diffeomorphism $\mathbf{CP}^1 \cong S^2$. (Hint: Use part (c) to construct a map $\mathbf{C}^2 \setminus \{0\} \to S^2$ with a certain property to apply theorem 4.30 of the textbook).