SOLUTIONS TO HOMEWORK 4

Problem 1.

(a) Such a section is $\sigma_0(x^1, \dots, x^n) = (x^1, \dots, x^n, 0, \dots, 0)$.

(b) Fix $p \in M$. Let $(\widetilde{V}, \psi)$, $(\widetilde{U}, \phi)$ be charts of M, N near $p, \pi(p)$ such that $\widehat{\pi} = \phi \circ \pi \circ \psi^{-1}$ has the coordinate representation as in the rank theorem. Without loss of generality we can assume that $\psi(p) = 0$ and $\phi(\pi(p)) = 0$. Since $\psi(\widetilde{V}) \subset \mathbf{R}^m$ is open there exists $\epsilon > 0$ such that the m-cube C_ϵ is contained in $\psi(\widetilde{V})$. Suppose $x = (x^1, \dots, x^m) \in C_\epsilon$, that is $|x^i| < \epsilon$ for all $i = 1, \dots, m$. Then $y = \pi(x) = (x^1, \dots, x^n)$ is in the cube C'_ϵ since each x^i has $|x^i| < \epsilon$ for $i = 1, \dots, n \leq m$. Conversely if $y = (y^1, \dots, y^n) \in C'_\epsilon$ then $x = (y^1, \dots, y^n, 0, \dots, 0) \in C_\epsilon$ and $\pi(x) = y$. This shows that $\widehat{\pi}(C_\epsilon) = C'_\epsilon$.

(c) Define $\widehat{\sigma}\colon C'_\epsilon \to C_\epsilon$ as the restriction of σ_0 from part (a) to the unit cube. Let $V = \psi^{-1}(C_\epsilon) \subset \widetilde{V}$ and $U = \phi^{-1}(C'_\epsilon) \subset \widetilde{U}$. Then the desired local section is

$$\sigma = \psi \circ \widehat{\sigma} \circ \phi^{-1} \colon U \to V. \tag{1}$$

(d) Suppose that for every p there admits such a local section σ. From the relation $\pi \circ \sigma = \mathbb{1}_U$ we have

$$d\pi_p \circ d\sigma_{\pi(p)} = \mathbb{1}_{T_{\pi(p)}U} \tag{2}$$

which implies that $d\pi_p$ is surjective.

Problem 2. Let (U, ϕ) be a chart for M. From this we associated a chart $(\pi^{-1}(U), \widetilde{\phi})$ for TM where $\widetilde{\phi}\colon \pi^{-1}(U) \to \mathbf{R}^n \times \mathbf{R}^n$ is

$$\widetilde{\phi}(p, v) = (\phi(p), d\phi_p(v)). \tag{3}$$

The coordinate representation of π with respect to these charts is very simple:

$$\widehat{\pi} = \phi \circ \pi \circ \widetilde{\phi}^{-1}(a, v) = a. \tag{4}$$

That is, if $\pi_1\colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ is the projection onto the first factor then $\widehat{\pi} = \pi_1$. Since π_1 is a submersion and ϕ is a local diffeomorphism it follows that π is also a submersion when restricted to $\pi^{-1}(U)$. Since the chart (U, ϕ) was arbitrary we are done.

Problem 3.

(a) Suppose that v is a vector in the tangent space at p of M viewed as a subspace of T_pN. That is $v = \mathrm{d}i_p(w)$ where $i\colon M \to N$ is the inclusion and w is some vector in T_pM. If $f \in C^\infty(N)$ is such that $f|_M = 0$ then

$$vf = \mathrm{d}i_p(w)f = w(f \circ i) = w(f|_M) = 0. \tag{5}$$

Conversely suppose $v \in T_pN$ satisfies $vf = 0$ for any $f \in C^\infty(N)$ whose restriction to M vanishes $f|_M = 0$. We want to argue that there exists $w \in T_pM$ such that $v = \mathrm{d}i_p(w)$. To do this we will choose slice coordinates for $M \subset N$ near $p \in M$. That is, we choose an open subset $U \subset N$ such that $U \cap M$ is the slice $x^{m+1} = \cdots = x^n = 0$, where the $(x^1, \ldots, x^m)$ are coordinates for $U \cap M$. With respect to these coordinates the inclusion is of the form

$$\widehat{i}(x^1, \ldots, x^m) = (x^1, \ldots, x^m, 0, \ldots, 0). \tag{6}$$

Now, the vector v can be written in these coordinates as

$$v = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}. \tag{7}$$

Furthermore, in these slice coordinates we see that the vectors $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^m}$ span T_pM, so that $v \in T_pM$ if and only if $v^i = 0$ for $i > m$.

To finish the proof we will show that $v^j = 0$ for any $j > k$. To do this we fix a bump function $\psi \in C^\infty(M)$ which is identically 1 on a small neighborhood of p and vanishes outside of U. For $j > k$, define $f^j \in C^\infty(M)$ by $f(x) = \psi(x)x^j$ for $x \in U$ (where we are using the coordinate on U) and $f \equiv 0$ outside of U. This is smooth by construction and since $\psi \equiv 1$ near p we have

$$\frac{\partial \psi(x) x^j}{\partial x^i}(p) = \delta^j_i. \tag{8}$$

Thus

$$0 = v(f) = \sum_i v^i \frac{\partial(\psi(x)x^j)}{\partial x^i}(p) = v^j. \tag{9}$$

as desired.

(b) Since $M \subset \mathbf{R}^n$ satisfies the local m-slice condition it follows that $TM \subset \mathrm{T}\mathbf{R}^n = \mathbf{R}^n \times \mathbf{R}^n$ satisfies the local $2m$-slice condition. Thus, $\mathrm{T}M$ is an embedded submanifold of $\mathbf{R}^{2n}$ of dimension $2m$.

Next, we show that UM is an embedded submanifold of $\mathrm{T}M$. Since the composition of embedded submanifolds is an embedded submanifold, this is sufficient. Define $f\colon \mathbf{R}^{2n} \to \mathbf{R}$ by the formula $(x, v) \mapsto |v|^2$. This is a smooth function and hence restricts to a smooth function $f|_{\mathrm{T}M}\colon \mathrm{T}M \to \mathbf{R}$.

Moreover, as a subset we have $UM = (f|_{\mathrm{T}M})^{-1}(1) \subset \mathrm{T}M$. We will apply the regular value theorem to conclude that $1 \in \mathbf{R}$ is a regular value.

The differential of the map f at any $(x, v) \in M$ is the linear map $\mathrm{d}f_{(x,v)} \colon \mathrm{T}_p(\mathrm{T}M) \to \mathbf{R}$ which sends a pair of m-vectors (u, w) to the number $2\sum_i v_i w_i$. So as long as $v \neq 0$ the differential at (x, v) has rank 1. The preimage $f^{-1}(1) = UM$ consists only of vectors with $v \neq 0$, so we are done.