HOMEWORK 5
DUE OCTOBER 13

There are two problems to turn in.

Problem 1.

(a) Notice that h_z, h_w are the restrictions of the globally defined function

$$h([z,w]) = \frac{|z|^2 - |w|^2}{|z|^2 + |w|^2}. \tag{1}$$

(b) Consider first the restriction of h to U_w. The complex coordinate for U_w is $\phi_w([z,1]) = z$, which in real coordinates is $\phi_w([x+\mathrm{i}y,1]) = (x,y)$. In this coordinate, the function h reads

$$\widehat{h}_w \overset{\text{def}}{=} h \circ \phi_w^{-1}(x,y) = h([x+\mathrm{i}y,1]) = \frac{x^2+y^2-1}{x^2+y^2+1}. \tag{2}$$

The differential of this function at (x,y) is represented by the 1×2 matrix

$$(\mathrm{d}h_w)_{(x,y)} = \left(\frac{4x}{(x^2+y^2+1)}, \frac{4y}{(x^2+y^2+1)} \right). \tag{3}$$

Thus, $\phi_w^{-1}(0,0) = [0,1]$ is the unique critical point contained in U_w. Notice that $h([0,1]) = -1$ so that -1 is a critical value of h.

Similarly $[1,0]$ is the only critical point of h contained in U_z and hence the only other critical value is $h([1,0]) = 1$.

(c) Suppose $-1 \leq c < 1$. Then $h^{-1}(c) \subset U_z$, and $h^{-1}(c)$ is the subset of points $[1,w] \in \mathbf{CP}^1$ such that

$$1 - |w|^2 = c\left(1 + |w|^2\right), \tag{4}$$

or equivalently

$$(c+1)|w|^2 = 1 - c. \tag{5}$$

If, in addition $c \neq -1$ then this can be written as $|w|^2 = \frac{1-c}{1+c}$. What this shows is that for $-1 < c < 1$ there is a diffeomorphism

$$f \colon h^{-1}(c) \to S^1, \tag{6}$$

where S^1 is the unit circle, defined by $f([1,w]) = \sqrt{\frac{1+c}{1-c}}w$.

(d) We have already shown that $h^{-1}(\pm 1)$ is the singleton set.

Problem 2. Define $\widetilde{f}\colon \mathbf{R}^n \setminus \{0\} \times \mathbf{R}_{>0}$ by $\widetilde{f}(x,\lambda) = f(\lambda x)$. One one hand, by chain rule, we have

$$\frac{\partial}{\partial \lambda}\widetilde{f}(x,\lambda) = \lambda^{-1}\sum_i x^i \frac{\partial}{\partial x^i} f(\lambda x). \tag{7}$$

On the other hand, since $\widetilde{f}(x,\lambda) = \lambda^c f(x)$ we have

$$\frac{\partial}{\partial \lambda}\widetilde{f}(x,\lambda) = c\lambda^{c-1} f(x). \tag{8}$$

Equating these two lines we obtain

$$\lambda^{c-1} E f = c\lambda^{c-1} f, \tag{9}$$

which implies the result.