HOMEWORK 6

Let $\mathbf{H}$ be the four-dimensional real vector space spanned by the vectors $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$. A ***quaternion*** is a vector in $\mathbf{H}$, so of the form

$$a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$$

where a, b, c, d are real numbers. Typically, we will omit the symbol $\mathbf{1}$ and simply write a quaternion as

$$a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}.$$

Recall that $\mathbf{C}$ is a real vector space with basis $\{1, \mathrm{i}\}$. Define the $\mathbf{R}$-linear map $\Phi\colon \mathbf{C} \times \mathbf{C} \to \mathbf{H}$ by

$$\Phi(1,0) = \mathbf{1}, \quad \Phi(\mathrm{i},0) = \mathbf{i}, \quad \Phi(0,1) = \mathbf{j}, \quad \Phi(0,\mathrm{i}) = -\mathbf{k}.$$

Define a multiplication on $\mathbf{C} \times \mathbf{C}$ by the rule

$$(z,w) \cdot (z',w') = (zz' - w'\overline{w}\ ,\ \overline{z}w' + z'w).$$

With this product, $\mathbf{C} \times \mathbf{C}$ is an algebra over $\mathbf{R}$.

(1) Using the isomorphism Φ we can transfer the multiplication on $\mathbf{C} \times \mathbf{C}$ to $\mathbf{H}$ to give it the structure of an algebra over $\mathbf{R}$. Show that with this multiplication the following relations hold

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$$

and

$$\mathbf{ijk} = -1.$$

Is $\mathbf{H}$ a commutative algebra?

(2) For $(a,b) \in \mathbf{C} \times \mathbf{C}$ define $(a,b)^* \stackrel{\text{def}}{=} (\overline{a}, -b) \in \mathbf{C} \times \mathbf{C}$. Via the isomorphism Φ this induces a $\mathbf{R}$-linear map $(-)^*\colon \mathbf{H} \to \mathbf{H}$. Show that the $\mathbf{R}$-bilinear operation

$$\langle -, - \rangle\colon \mathbf{H} \times \mathbf{H} \to \mathbf{H}$$

defined by $\langle p, q \rangle = \frac{1}{2}(p^*q + q^*p)$ is a real inner product.

(3) Suppose $p \in \mathbf{H}$ is a nonzero vector. Show that the element

$$p^{-1} \stackrel{\text{def}}{=} \langle p, p \rangle^{-1} p^*$$

is a two-sided inverse for p.

(4) Show that $\mathbf{H}^{\times}$ (the set of nonzero quaternions) has the structure of a four-dimensional Lie group.

(5) Let $\mathbf{S} \subset \mathbf{H}^\times$ be the set of unit quaternions. That is, the set of vectors $p \in \mathbf{H}^\times$ such that $\langle p, p \rangle = 1$. Show that $\mathbf{S}$ is an embedded Lie subgroup of $\mathbf{H}^\times$ which is diffeomorphic to S^3.

(6) Suppose $p \in \mathbf{H}$ is of the form $p = b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ (that is, p has no component in the $\mathbf{1}$ direction). Show that for every $q \in \mathbf{S}$ that qp is tangent to $\mathbf{S}$ at q.

(7) Show that the three vector fields $X, Y, Z \in \mathrm{Vect}(\mathbf{H})$ defined by

$$X|_q = q\mathrm{i}, \quad Y|_q = q\mathbf{j}, \quad Z|_q = q\mathbf{k}$$

restrict to a global frame for $\mathbf{S}$. (Bonus: Show that this frame is left-invariant).