HOMEWORK 6
SOLUTIONS

(1) This is immediate verification. For example

$$\mathbf{ijk} = \Phi\left((\mathrm{i},0)\cdot(0,1)\cdot(0,-\mathrm{i})\right) = -\Phi(1,0) = -1. \tag{1}$$

(2) Observe that $\langle -,- \rangle$ is bilinear and symmetric. We check that $\langle p,q \rangle \in \mathbf{R}\cdot\mathbf{1} = \mathbf{R}$ for any $p,q \in \mathbf{R}$. This will follow from the lemma.

Lemma 0.1. *$(pq)^* = q^*p^*$ for any $p,q \in \mathbf{H}$. Moreover, $x = x^*$ if and only if $x = \lambda\mathbf{1}$ for some $\lambda \in \mathbf{R}$.*

Proof. Let $p = \Phi(z,w), q = \Phi(z',w')$. Then $(pq)^*$ is the image, under Φ, of

$$(zz' - w'\overline{w}, \overline{z}w' + z'w)^* = (\overline{z}\overline{z}' - \overline{w}'w, -\overline{z}w' - \overline{z}'w). \tag{2}$$

On the other hand q^*p^* is the image, under Φ, of

$$(\overline{z}', -w')\cdot(\overline{z}, -w) \tag{3}$$

which agrees with the formula above. The second assertion is also a direct verification. □

By the lemma $\langle p,q \rangle^* = \frac{1}{2}(q * p + pq^*) = \langle p,q \rangle$ so that $\langle p,q \rangle \in \mathbf{R1}$ as desired. Finally, suppose that $\Phi(z,w) = p$. Then $\langle p,p \rangle = \Phi(|z|^2 + |w|^2, 0) > 0$ so long as $p \neq 0$.

(3) We have

$$pp^{-1} = p\langle p,p \rangle^{-1}p^* = \langle p,p \rangle\langle p,p \rangle^{-1} = 1. \tag{4}$$

Similarly, $p^{-1}p = 1$.

(4) The multiplication $\mathbf{H}^\times \times \mathbf{H}^* \to \mathbf{H}^\times$ is the restriction of a smooth map $\mathbf{H} \times \mathbf{H} \to \mathbf{H}$ and is hence smooth. Similarly $(-)^{-1} \colon \mathbf{H}^\times \to \mathbf{H}^\times$ is smooth.

(5) $\mathbf{S}$ is a level set of the smooth submersion $\mathbf{H}^\times \to \mathbf{R}$ which sends $p \mapsto \langle p,p \rangle$. Identifying $\mathbf{H}^\times \cong \mathbf{R}^4 \setminus \{0\}$, this map is the usual norm on four-vectors and the result follows.

(6) This follows from the following lemma.

Lemma 0.2. *Let $q \in \mathbf{S}$. An element $v \in \mathbf{H} = T_q\mathbf{H}$ is in $T_q\mathbf{S}$ if and only if $\langle v,q \rangle = 0$.*

Proof. Recall the characterization of the tangent space of a submanifold from HW 4. Applied to this situation we see that $T_q\mathbf{S}$ is the set of vectors $v \in \mathbf{H}$ such that $vf = 0$ whenever $f|_\mathbf{S} = 0$. Suppose that $v \in T_q\mathbf{S}$ and consider the function $f \in C^\infty(\mathbf{H})$ defined by $f(p) = \langle p,p \rangle - 1$. Clearly $f|_\mathbf{S} = 0$.

Moreover $vf = 2\langle q, v\rangle$ which implies $\langle q, v\rangle = 0$. This shows that $T_q\mathbf{S} \subset \{v \in \mathbf{H} \mid \langle v, q\rangle = 0\}$. To see the reverse inclusion note that each set is a 3-dimensional linear subspace of $\mathbf{H}$. □

To complete the problem we show that $\langle qp, q\rangle = 0$. Suppose $p = \mathbf{i}$, then

$$\langle q\mathbf{i}, q\rangle = \frac{1}{2}\left((q\mathbf{i})^* q + q^* q\mathbf{i}\right) = 0 \tag{5}$$

since $\mathbf{i}^* = -\mathbf{i}$. Similarly $\langle q\mathbf{j}, q\rangle = \langle q\mathbf{k}, q\rangle = 0$.

(7) By the previous problem we see that X, Y, Z admit well-defined restrictions to $\mathbf{S}$. Observe that $q\mathbf{i}, q\mathbf{j}, q\mathbf{k}$ are linearly independent whenever $q \neq 0$, this shows that $X|_{\mathbf{S}}, Y|_{\mathbf{S}}, Z|_{\mathbf{S}}$ provide a frame for $\mathbf{S}$.

Observe that $L_q\colon \mathbf{S} \to \mathbf{S}$ is the restriction of the linear map $\mathbf{H} \to \mathbf{H}$ which is $p \mapsto qp$. In particular, $(\mathrm{d}L_q)_{q'} = L_q$ for every $q, q' \in \mathbf{S}$. Applying this, we see that

$$((L_q)_* X)_{q'} = (\mathrm{d}L_q)_{q'} X_{q'} = (\mathrm{d}L_q)_{q'}(q'\mathrm{i}) = qq'\mathrm{i} = X_{qq'}. \tag{6}$$

Similarly, Y, Z are left invariant.