HOMEWORK 7
DUE OCTOBER 27

There are two problems to turn in.

(1) Let G be a Lie group and M a smooth manifold. A ***left*** ***G-action on*** M is a smooth map

$$\rho\colon G \times M \to M$$

such that $\rho(gh, p) = \rho(g, \rho(h, p))$ and $\rho(e, p) = p$ for all $g, h \in G$ and $p \in M$. Let $\mathfrak{g} = \mathrm{Lie}(G)$.

(a) Fix $p \in M$ and consider the map $\rho_p \overset{\mathrm{def}}{=} \rho(-, p)\colon G \to M$. Show that for each $x \in \mathfrak{g}$ the assignment

$$p \in M \mapsto -(\mathrm{d}\rho_p)_e(x)$$

defines a vector field on M that we denote $X^\rho(x)$.

(b) Show that the assignment

$$x \in \mathfrak{g} \mapsto X^\rho(x)$$

defines a homomorphism of Lie algebras $X^\rho\colon \mathfrak{g} \to \mathrm{Vect}(M)$. (A homomorphism of Lie algebras $\phi\colon \mathfrak{g} \to \mathfrak{h}$ is a linear map such that $\phi([x, y]) = [\phi(x), \phi(y)]$.)

(c) Suppose that the left G-action ρ is **free**, meaning that for any $p \in M$ one has $\rho(g, p) = p$ if and only if $g = e$. Show that X^ρ is injective.

(2) Recall that $SL(n, \mathbf{R})$ is the Lie group of $n \times n$ matrices with determinant equal to 1.

(a) In class we showed that $SL(n, \mathbf{R})$ is a smooth submanifold of the vector space of $n \times n$ matrices. In particular, for any $X \in SL(n, \mathbf{R})$ we can view $T_X SL(n, \mathbf{R})$ as a linear subspace of $Mat_n(\mathbf{R})$. Show that if $A \in T_{\mathbb{1}} SL(n, \mathbf{R})$ then $\mathrm{tr}(A) = 0$.

(b) Let

$$\mathfrak{sl}(n, \mathbf{R}) = \{A \in \mathrm{Mat}_n(\mathbf{R}) \mid \mathrm{tr}(A) = 0\}.$$

Show that $\mathfrak{sl}(n, \mathbf{R})$ is a Lie subalgebra of $\mathfrak{gl}(n, \mathbf{R})$.

(c) Construct a Lie algebra isomorphism

$$\varepsilon\colon \mathrm{Lie}(SL(n, \mathbf{R})) \to \mathfrak{sl}(n, \mathbf{R}).$$