HOMEWORK 9
SOLUTIONS

There was a small typo in the original problem: the equivalence relation should read

$$(x, y) \sim (x + n, (-1)^n y) \tag{1}$$

for some integer $n \in \mathbf{Z}$. More properly, this equivalence relation is $\mathbf{Z}$-generated by $(x, y) \sim (x + 1, -y)$.

(1) Recall that $S^1 = \mathbf{R}/\sim$ where $\sim$ is the equivalence relation $x \sim x + n$. Define $\pi([x, y]) = [x]$.

(2) First we decide on a topology on E. Declare $U \subset E$ is open if and only if $q^{-1}(U) \subset \mathbf{R}^2$ is open. Next, we need to construct a smooth atlas on E. Let

$$\widetilde{U}_1 = \{(x, y) \mid x \in (0, 1), y \in \mathbf{R}\} \subset E \tag{2}$$

Then $U_1 = q(\widetilde{U}_1) \subset E$ is open. Define

$$\phi_1 \colon U_1 \to \mathbf{R}^2 \tag{3}$$

by the formula $\phi_1([x, y]) = (x, y)$. To see that this is well-defined, suppose that $[x, y] = [x', y']$ in U_1. Then $x' = x + n$ and $y' = (-1)^n y$ for some integer n. But one has $(x + n, (-1)^n y) \in \widetilde{U}_1$ if and only if $n = 0$. Thus, ϕ_1 is well-defined. It is immediate to see that ϕ_1 is a homeomorphism onto its image. Similarly, let

$$\widetilde{U}_2 = \{(x, y) \mid x \in (-1/2, 1/2), y \in \mathbf{R}\} \subset \mathbf{R}^2, \tag{4}$$

then $U_2 \stackrel{\text{def}}{=} q(\widetilde{U}_2)$ is open. Define

$$\phi_2 \colon U_2 \to \mathbf{R}^2 \tag{5}$$

by the formula $\phi_2([x, y]) = (x + 1, y)$. Similarly as before, we see that ϕ_2 is well-defined and a homeomorphism onto its image.

It remains to see that $\phi_1 \circ \phi_2^{-1}$ is smooth. Note that

$$U_1 \cap U_2 = q((0, 1/2) \times \mathbf{R}) \cup q((1/2, 1) \times \mathbf{R}). \tag{6}$$

Thus $\phi_1(U_1 \cap U_2) = \phi_2(U_1 \cap U_2)$ is the disjoint union

$$\{(x, y) \mid x \in (0, 1/2), y \in \mathbf{R}\} \cup \{(x, y) \mid x \in (1/2, 1), y \in \mathbf{R}\} = W_1 \cup W_2. \tag{7}$$

On W_1 we have $\phi_1 \circ \phi_2^{-1}(x, y) = (x, y)$ and on W_2 we have $\phi_1 \circ \phi_2^{-1}(x, y) = (x, -y)$.

In each of the coordinates ϕ_1, ϕ_2 the map π is simply the projection $(x, y) \mapsto x$, which is a submersion.

(3) Let $\widetilde{V}_1 = (0, 1) \subset \mathbf{R}$ and $V_1 = \exp(V_1) \subset S^1$; $\widetilde{V}_2 = (-1/2, 1/2)$ and $V_2 = \exp(V_2) \subset S^1$. We construct local trivializations over the open sets V_1, V_2. Notice that $\pi^{-1}(V_i) = U_i$ for $i = 1, 2$. We define the local trivializations

$$\psi_i \colon U_i \to V_i \times \mathbf{R} \tag{8}$$

for $i = 1, 2$ by the formula $\psi_i([x, y]) = ([x], y)$.

The transition function is $g_{12} \colon V_1 \cap V_2 \to \mathbf{R}^\times$ satisfying

$$\psi_1 \circ \psi_2^{-1}([x], a) = ([x], g_{12}([x])a). \tag{9}$$

Note that $V_1 \cap V_2$ is the disjoint union of the northern and souther hemispheres of the circle. On the northern hemisphere one has $g_{12}([x]) = 1$ and on the southern one has $g_{12}([x]) = -1$. Since g_{12} is not constant we see that E is not the trivial bundle. (In fact, E is not even isomorphic to the trivial bundle.)