Case 1: Sps $p \in R(V)$. Then by the rank thm we can find coordinates near p s.t.

$$V = \frac{\partial}{\partial x^1} \quad \text{in local coords.}$$

Then the corresponding flow is

$$\theta_t(x) = (t + x^1, x^2, \dots, x^n).$$

$\Rightarrow$

$$(d\theta_{-t})_{\theta_t(x)} X_{\theta_t(x)} =$$

$$(d\theta_{-t})_{\theta_t(x)} \left(X^j(x^1+t, x^2, \dots, x^n) \frac{\partial}{\partial x^j}\Big|_{\theta_t(x)} \right)$$

$$= X^j(x^1+t, x^2, \dots, x^n) \frac{\partial}{\partial x^j}\Big|_x .$$

November 1

$$\begin{array}{c} E \\ \downarrow \pi \\ M \end{array} \quad \text{vector bundle.}$$

The most important axiom of a vector bundle is <u>local triviality</u>. This says that every point $p \in M$ has a nbd

$$U \subset M$$

and

$$\psi : \underset{\substack{\| \\ E|_U}}{\pi^{-1}(U)} \xrightarrow{\cong} U \times \mathbb{R}^k$$

diffeomorphism

s.t.

1) $p_1 \circ \psi = \pi \big|_{E|_U}$.

2) For every $q \in U$ ψ determines a linear isomorphism

$$\psi\big|_{E_q} : E_q \xrightarrow{\cong} \{q\} \times \mathbb{R}^k.$$

Prop: $\pi : TM \to M$ is a rank $= \dim M$ smooth vector bundle.

Pf: Given chart (U, ϕ) of M we have trivializations

$$\psi : \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^k$$

$$\rotatebox{90}{$\in$}$$

$$v^i \frac{\partial}{\partial x^i}\Big|_p \longmapsto \left(p, (v^i)\right).$$

This is linear on fibers and satisfies $p_1 \circ \psi = \pi$. The composition

$$\pi^{-1}(U) \xrightarrow{\psi} U \times \mathbb{R}^n \xrightarrow{\phi \times \mathbb{1}} \phi(U) \times \mathbb{R}^n$$

is the chart for $\pi^{-1}(U)$. So ψ is a diffeomorphism. $\blacksquare$

Lem: Sps $E \xrightarrow{\pi} M$ is rk k smooth v.b., and take two trivializations

$$\psi : \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^k$$

$$\chi : \pi^{-1}(V) \longrightarrow V \times \mathbb{R}^k .$$

There exists a unique smooth map

$$g_{UV} : U \cap V \longrightarrow GL(k)$$

s.t. $\psi \circ \chi^{-1}(p, v) = (p, g_{UV}(p)\, v)$.

Pf: Have commuting diagram:

$$\begin{tikzcd}
(U \cap V) \times \mathbb{R}^k \arrow[dr, "p_1"'] & \pi^{-1}(U \cap V) \arrow[l, "\chi"'] \arrow[r, "\psi"] \arrow[d, "\pi"] & (U \cap V) \times \mathbb{R}^k \arrow[dl, "p_1"] \\
& U \cap V &
\end{tikzcd}$$

$\Rightarrow \quad p_1 \circ (\psi \circ \chi^{-1}) = p_1 . \quad \Rightarrow$

$$\psi \circ \chi^{-1}(p, v) = (p, \sigma(p, v))$$

for some smooth $\sigma : (U \cap V) \times \mathbb{R}^k \to \mathbb{R}^k$.

But, for each $p \in U \cap V$ the map

$$v \longmapsto \sigma(p, v)$$

is a bijective linear map. Hence $\exists$
$g_{UV}(p) \in GL(k)$ s.t.

$$\sigma(p, v) = g_{UV}(p)\, v.$$

Remains to show

$$p \longmapsto g_{UV}(p)$$

is smooth. For this, let $\{e_i\}$ be a basis for $\mathbb{R}^k$. Then

$$\begin{aligned} U \cap V &\longrightarrow \mathbb{R} \\ p &\longmapsto g_{UV}(p)^i_{\ j} \end{aligned}$$

is the composition

$$U \cap V \longrightarrow (U \cap V) \times \mathbb{R}^n \xrightarrow{\sigma} \mathbb{R}^k \longrightarrow \mathbb{R}$$

$$p \longmapsto (p, e^i) \qquad v \longmapsto v_j .$$

$\boxed{\kappa}$

The maps

$$g_{UV} : U \cap V \longrightarrow GL(k, \mathbb{R})$$

is called the transition fn for the local trivializations ψ, χ. In the sense that we explain now, they completely determine the data of the vector bundle.

Dfn: Let

$$M \longrightarrow \mathrm{Vect}(k)$$
$$p \longmapsto E_p$$

be an assignment of a vector space of dim $= k$ for each $p \in M$. Let $E = \coprod_p E_p$.

Gluing data for $E = \coprod_p E_p$ is:

1) An open cover $\{\mathcal{U}_\alpha\}$ of M.

2) For each α, a bijection

$$\psi_\alpha : \pi^{-1}(\mathcal{U}_\alpha) \longrightarrow \mathcal{U}_\alpha \times \mathbb{R}^k$$

s.t. for each $p \in \mathcal{U}_\alpha$ have linear iso:

$$\psi_\alpha\big|_{E_p} : E_p \xrightarrow{\cong} \{p\} \times \mathbb{R}^k .$$

3) For α, β w/ $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \phi$, a smooth map

$$g_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \longrightarrow GL(k, \mathbb{R})$$

s.t.

$$\psi_\alpha \circ \psi_\beta^{-1}(p, v) = (p, g_{\alpha\beta}(p) v) .$$

Prop: If $E = \sqcup_p E_p$ as above, then gluing data for E determines the unique structure of a vector bundle on

$$\pi : E \longrightarrow M.$$

Pf: Next time.

Ex: Sps E, E' are v.b's of rank k, k' respectively. The Whitney sum is the v.b.

$$E'' = E \oplus E'$$

where - $(E \oplus E')_p = E_p \oplus E_p'$

- $\pi'' : E'' \to M$, $(v_q, v_p') \longmapsto p$.

- If $p \in M$ choose $U \subset M$ nbd of p small enough so that we have local trivializations for

$E|_U$, $E'|_U$:

$$\psi : \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^k$$

$$\psi' : \pi'^{-1}(U) \longrightarrow U \times \mathbb{R}^{k'}.$$

The trivialization for $E \oplus E'|_U$ is

$$\psi'' : \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^{k+k'}$$

$$\psi''(v, v') =$$

$$\left(\pi(v), \left(\pi_{\mathbb{R}^k} \circ \psi(v), \pi_{\mathbb{R}^{k'}} \circ \psi'(v')\right)\right).$$

If $U \cap V \neq \phi$ the transition fn's for $E \oplus E'|_{U \cap V}$ is

$$g''_{UV} = \begin{pmatrix} g_{UV} & 0 \\ 0 & g'_{UV} \end{pmatrix}.$$