$f = 1_M$. That is

$$\begin{tikzcd} E \arrow[rr, "F"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\ & M & \end{tikzcd}$$

s.t. $F|_{E_p} : E_p \to E'_p$ is liner.

November 10

- A <u>subbundle</u> $\begin{matrix} D \subset & E \\ & \downarrow \\ & M \end{matrix}$

is a v.b. $\begin{matrix} D \\ \downarrow \pi_D \\ M \end{matrix}$ s.t. 1) $D \subset E$

is a submanifold, 2) $\pi_D = \pi_E|_D$.

and 3) $D_p = D \cap E_p < E_p$ is a

liner subspace, and the vector space

str. on D_p is inherited from E_p.

Some examples.

- If $f: M \to N$ is smooth then

$$df : TM \longrightarrow TN$$

is a bundle homomorphism.

- If $\begin{matrix} E \\ \downarrow \pi \\ M \end{matrix}$ is a v.b. and $S \subset M$ is a submanifold, then we have a bundle homomorphism

$$\begin{array}{ccc} E|_S & \longrightarrow & E \\ \downarrow & & \downarrow \\ S & \longrightarrow & M \end{array}$$

- Cross product w/ a fixed vector field Z on $\mathbb{R}^3$ is a bundle map

$$Z \times (-) : T\mathbb{R}^3 \longrightarrow T\mathbb{R}^3.$$

· A bundle homomorphism $F : E \to E'$ over M

$$\begin{array}{ccc} E & \xrightarrow{F} & E' \\ & \searrow \; M \; \swarrow & \end{array}$$

induces

$$\tilde{F} : \Gamma(M, E) \longrightarrow \Gamma(M, E')$$

by the formula

$$(\tilde{F} s)(p) = F(s(p)) .$$

The map $\tilde{F}$ is linear. But more is true.

Given any $s \in \Gamma(M, E)$ and $f \in C^\infty(M)$ we define a new section $fs \in \Gamma(M, E)$ by

$$(fs)(p) = f(p) \cdot s(p).$$

mult in fibers E_p.

In this sense $\Gamma(M, E)$ is a $C^\infty(M)$-module. The map $\tilde{F}$ is a homomorphism of $C^\infty(M)$-modules:

$$\tilde{F}(fs) = f \cdot \tilde{F}(s).$$

for all $f \in C^\infty(M)$, $s \in \Gamma(M, E)$.

- Sps $F : E \to E'$ is a smooth bundle homomorphism over M. Define

$$\ker(F) = \bigcup_{p \in M} \ker\left(F|_{E_p}\right) \subset E$$

$$\operatorname{im}(F) = \bigcup_{p \in M} \operatorname{im}\left(F|_{E_p}\right) \subset E'.$$

<u>Thm</u>: $\ker F$, $\operatorname{im} F$ are subbundles of E, E', iff

$$q \longmapsto \operatorname{rank}\left(F|_{E_p}\right)$$

is constant.

Pf: If $\ker F$, $\operatorname{im} F$ are subbundles, then clearly F must have constant rank.

Sps F has constant rank. We will show that ker/im is equipped w/ local trivializations. For the full proof that $\ker F / \operatorname{im} F$ is subbundle, see Lee, Thm 10.34.

Let $p \in M$ and pick nbd $U \ni p$ which admits a local frame

$$\{s_1, \ldots, s_n\} \text{ for } E|_U .$$

For each i, $F \circ s_i : U \to E'$ is a local section of E'. Wlog, we can assume

$$\{F \circ s_1(p), \ldots, F \circ s_r(p)\}, \quad r = \operatorname{rank}(F|_{E_p}).$$

form a basis for $\operatorname{im}(F|_{E_p})$.

By continuity, these remain linearly independent in some nbd $\mathcal{U}_0$ of p.

$\rightsquigarrow$ get local frame for $\mathrm{im}(F)$.

Similarly for ker .... $\boxed{///}$

Ex: The condition of constant rank is necessary!
Consider the trivial rk 1 bundle over $\mathbb{R}$:

$$\begin{array}{c} E = \mathbb{R} \times \mathbb{R} \\ \downarrow \\ \mathbb{R} \end{array}$$

Define the bundle map

$$F : E \longrightarrow E'$$
$$(x, t) \longmapsto (x, xt).$$

Then $\mathrm{rk}\left(F|_{E_x}\right) = \begin{cases} 1 & , \ x \neq 0 \\ 0 & , \ x = 0 \end{cases}.$

And hence

$$\ker(F|_{E_x}) \cong \begin{cases} 0, & x \neq 0 \\ \mathbb{R}, & x = 0. \end{cases}$$

Not a vector bundle str on $\bigcup_x \ker(F|_{E_x})$.