October 10

Let $\pi : A \to B$ be a map of sets.
A <u>section</u> of π is a map $\sigma : B \to A$ s.t. $\pi \circ \sigma = \mathbb{1}_B$.

$$\begin{tikzcd} A \arrow[d, "\pi"'] \\ B \arrow[u, bend right, dashed, "\sigma"'] \end{tikzcd}$$

<u>Dfn</u>: Let M be smooth manifold. A <u>vector field</u> on M is a smooth section of $\pi : TM \to M$:

$$\begin{tikzcd} TM \arrow[d, "\pi"'] \\ M \arrow[u, bend right, dashed, "X"'] \end{tikzcd} \qquad \pi \circ X = \mathbb{1}_M.$$

In other words, for each $p \in M$ a vector field assigns a vector in the tangent space:

$$X_p \in T_p M.$$

If we choose coordinates on some $U \underset{\text{open}}{\subset} M$, then a vector field can locally be expressed as

$$X\Big|_U = \sum_{i=1}^n X^i(x) \frac{\partial}{\partial x^i}$$

where $X^i \in C^\infty(U)$, $i=1,\dots,n$.

<u>Prop</u>: A section X of π is smooth $\iff$ $\forall$ coordinates X^i is smooth $i=1,\dots,n$.

<u>Ex</u>: On $U \subset \mathbb{R}^n$ we have global coordinate, so every v.f. is of the form

$$X = \sum X^i(x) \frac{\partial}{\partial x^i}$$

for $X^i \in C^\infty(U)$.

<u>Ex</u>: Any proper open $U \subsetneq S^1$ admits an angle coordinate

$$\theta : U \longrightarrow \mathbb{R}.$$

Another angle coordinate $\tilde{\theta}$ differs from θ by a constant. So

$$\frac{\partial}{\partial \theta} = \frac{\partial}{\partial \tilde{\theta}}.$$

$\Rightarrow$ $\frac{\partial}{\partial \theta}$ defines a v.f. on S^1.

- We will write

$$\mathrm{Vect}(M) = \left\{ \begin{array}{c} \text{set of all} \\ \text{v.f.'s on } M \end{array} \right\}.$$

<u>Lem</u>: The operations

1) $(\lambda X)(p) = \lambda X(p)$

scalar mult. in T_pM

2) $(X+Y)(p) = X(p) + Y(p)$

addition in T_pM

endow $\mathrm{Vect}(M)$ w/ the structure of a vector space.

- Actually, more operations! If $f \in C^\infty(M)$ and $X \in \mathrm{Vect}(M)$ then we obtain a v.f.

$$fX \in \mathrm{Vect}(M)$$

defined by $(fX)(p) = f(p)\,X(p)$.

$\Rightarrow$ $\mathrm{Vect}(M)$ is a module over the ring $C^\infty(M)$. (Check the details)

• Let $U \subset M^n$ be open. A <u>frame</u> for U is a collection of v.f.'s

$$(X_1, \dots, X_n) \in \mathrm{Vect}(U)$$

s.t. $\{X_1(p), \dots, X_n(p)\}$ span $T_p M$

for every $p \in U$.

<u>Lemma</u>: Sps (U, ϕ) is a chart for M, and let $\{x^i\}$ be the associated coordinate. Then the v.f.'s

$$\left\{ \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \right\} \text{ are a frame for } U.$$

Pf: Here $\frac{\partial}{\partial x^i}$ is the v.f. which is

$$U \ni p \longmapsto \left. \frac{\partial}{\partial x^i} \right|_p .$$

By defn $\left\{ \frac{\partial}{\partial x^i}\Big|_p \right\}$ are linearly independent for each $p \in \mathcal{U}$. $\boxed{///}$

- A <u>global frame</u> is a frame for $\mathcal{U} = M$.

<u>Ex</u>: • $\mathbb{R}^n$, S^1 both admit global frames.

• S^2 does not admit a global frame.

If there exists a global frame for M we say that M is <u>parallelizable</u>. It turns out that the <u>only</u> parallelizable spheres are

$$S^1, S^3, \text{ and } S^7.$$