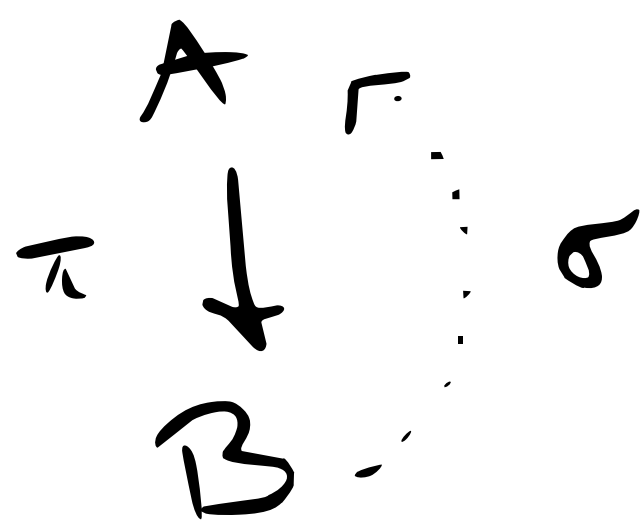


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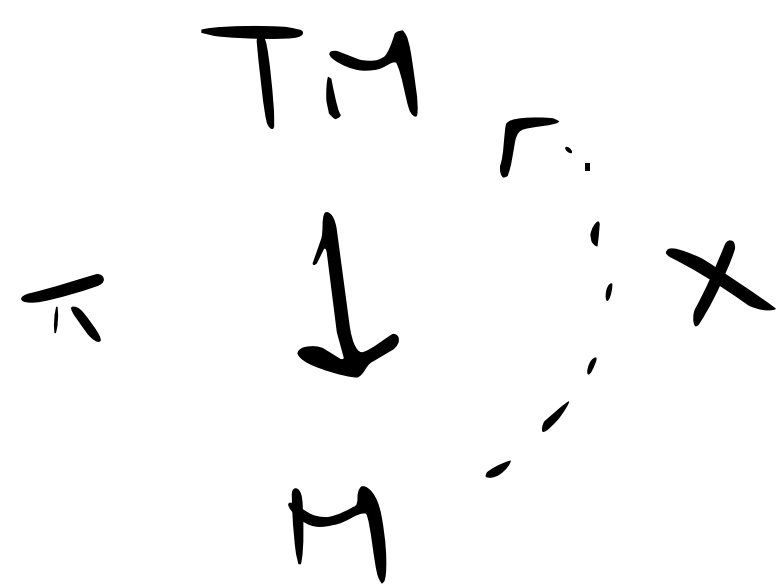
Let  $\pi : A \rightarrow B$  be a map of sets.

A section of  $\pi$  is a map  $\sigma : B \rightarrow A$

s.t.  $\pi \circ \sigma = \mathbb{1}_B$ .



Dfn: Let  $M$  be smooth manifold. A vector field on  $M$  is a smooth section of  $\pi : TM \rightarrow M$ :



$$\pi \circ X = \mathbb{1}_M.$$

In other words, for each  $p \in M$  a vector field assigns a vector in the tangent space:

$$X_p \in T_p M.$$

If we choose coordinates on some  $U \subset M$ ,  
then a vector field can locally be  
expressed as

$$X|_U = \sum_{i=1}^n x^i(x) \frac{\partial}{\partial x^i}$$

where  $x^i \in C^\infty(U)$ ,  $i=1, \dots, n$ .

Prop: A section  $X$  of  $\pi$  is smooth  
( $\Rightarrow$ )  $\forall$  coordinates  $x^i$  is smooth  $i=1, \dots, n$ .

Ex: On  $U \subset \mathbb{R}^n$  we have global coordinate,  
so every v.f. is of the form

$$X = \sum x^i(x) \frac{\partial}{\partial x^i}$$

for  $x^i \in C^\infty(U)$ .

Ex: Any proper open  $U \subset S'$  admits an angle coordinate

$$\theta : U \rightarrow \mathbb{R}$$

Another angle coordinate  $\tilde{\theta}$  differs from  $\theta$  by a constant. So

$$\frac{\partial}{\partial \theta} = \frac{\partial}{\partial \tilde{\theta}}$$

$\Rightarrow \frac{\partial}{\partial \theta}$  defines a v.f. on  $S'$ .

• We will write

$$\text{Vect}(M) = \left\{ \begin{array}{l} \text{set of all} \\ \text{v.f.'s on } M \end{array} \right\}.$$

Lemma: The operations

$$1) (\lambda X)(p) = \lambda X(p)$$

↖ scalar mult. in  
 $T_p M$

$$2) (X+Y)(p) = X(p) + Y(p)$$

↖ addition in  $T_p M$

endow  $\text{Vect}(M)$  w/ the structure of a  
vector space.

• Actually, more operations! If  $f \in C^\infty(M)$   
and  $X \in \text{Vect}(M)$  then we obtain a  
v.f.

$$fX \in \text{Vect}(M)$$

defined by  $(fX)(p) = f(p)X(p)$ .

$\Rightarrow \text{Vect}(M)$  is a module over the  
ring  $C^\infty(M)$ . (Check the details)

• Let  $U \subset \mathbb{R}^n$  be open. A frame for  $U$  is a collection of v.f.'s

$$(X_1, \dots, X_n) \in \text{Vect}(U)$$

$$\text{s.t. } \{X_1(p), \dots, X_n(p)\} \text{ span } T_p \mathbb{R}^n$$

for every  $p \in U$ .

Lemma:  $\text{Sp}_p(U, \phi)$  is a chart for  $\mathbb{R}^n$ ,

and let  $\{x^i\}$  be the associated coordinates.

Then the v.f.'s

$$\left\{ \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \right\} \text{ are a frame for } U.$$

PP: Here  $\frac{\partial}{\partial x^i}$  is the v.f. which is

$$U \ni p \longmapsto \frac{\partial}{\partial x^i} \Big|_p.$$

By defn  $\left\{ \frac{\partial}{\partial x^i} \Big|_p \right\}$  are linearly independent  
for each  $p \in U$ . □

• A global frame is a frame for  $U = M$ .

Ex: •  $\mathbb{R}^n$ ,  $S^1$  both admit global frames.

•  $S^2$  does not admit a global frame.

If there exists a global frame for  $M$   
we say that  $M$  is parallelizable. It  
turns out that the only parallelizable spheres  
are

$S^1$ ,  $S^3$ , and  $S^7$ .