October 10

Recall $T_pM = \left\{ \begin{array}{c} \text{Derivations at } p \\ C^\infty(M) \xrightarrow{v} \mathbb{R} \end{array} \right\}$

In particular, a v.f. $X$ on $M$ is an assignment of a derivation $X_p$ for each $p \in M$.

Given $f \in C^\infty(M)$ define

$$Xf \in C^\infty(M)$$

by $$(Xf)(p) = X_p(f).$$

**Prop**: Let $X : M \to TM$ be a section of $\pi : TM \to M$. TFAE:

1) $X$ is a smooth v.f.
2) $\forall f \in C^\infty(M)$, $Xf \in C^\infty(M)$
3) For each $U \subseteq_{op} M$, $f \in C^\infty(U)$, then $Xf$ is smooth on $U$.

Pf: 1) $\Rightarrow$ 2). Sps $X$ is smooth, let $f \in C^\infty(M)$. Then in coordinates

$$Xf(x) = \sum_{i=1}^{n} X^i(x) \frac{\partial f}{\partial x_i}(x).$$

Since $X^i(x)$ are smooth so is $Xf(x)$.

2) $\Rightarrow$ 3) Sps $U \subset M$ is open, $f \in C^\infty(U)$. For $p \in M$, let $\psi$ be a bump function

1) $\psi \in C^\infty(M)$

2) $V \subset U$ contains $p$ and $\psi|_V \equiv 1$.

3) $\psi|_{M-U} \equiv 0$.

Then define $\tilde{f} = \psi f$, which is defined to be $0$ away from $U$.

The $X\tilde{f}$ is smooth by assumption.

And
$$X\tilde{f}\Big|_V = Xf\Big|_V .$$

$\Rightarrow$ $Xf$ is smooth on a nbd of each pt of $U$.

3) $\Rightarrow$ 1). Take a possibly smaller nbd where there is a chart. On this nbd:
$$X(x^i) = \sum_j X^j \frac{\partial}{\partial x^j} = X^i .$$

$\Rightarrow$ $X^i$ smooth for each $i$. ▨

So $X \in \mathrm{Vect}(M)$ defines
$$X : C^\infty(M) \longrightarrow C^\infty(M)$$
$$f \longmapsto Xf .$$

[$R$ a ring. A $R$-derivation (or just a derivation) is a linear map

$$D : R \longrightarrow R$$

s.t. $D(ab) = Da \cdot b + a \cdot Db$. ]

**Prop**: $\mathrm{Vect}(M) \cong \mathrm{Der}(C^\infty(M))$.

vector space of all derivations.

**Pf**: Being a derivation at each pt implies

$$X(fg) = (Xf)\cdot g + f\cdot(Xg).$$

Conversely, sps $D \in \mathrm{Der}\, C^\infty(M)$. We will produce a v.f. $X$ s.t.

$$Df = Xf \quad \forall f \in C^\infty(M).$$

Define for each $p \in M$ $X_p \in T_p M$ by

$$X_p f = (Df)(p).$$

(Since $D$ is derivation it follows that $X_p$ is a derivation at $p$.)

Now since $Df$ is smooth $\Rightarrow$ $p \mapsto X_p$ is smooth.

$\square$

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Let $F : M \to N$ be smooth. Recall that for each $p \in M$ we have

$$dF_p : T_p M \longrightarrow T_p N.$$

**Q**: Does this define

$$\text{"} dF : \mathrm{Vect}(M) \to \mathrm{Vect}(N) \text{"} ?$$

In general, no. For example, if $F$ is not surjective then we do not know how to define "$dF(X)$" as a v.f. on $N$. Similar issue if $F$ is not injective.

**Dfn**: $X \in \mathrm{Vect}(M)$ and $Y \in \mathrm{Vect}(N)$ are $F$-related if

$$dF_p(X_p) = Y_{F(p)}$$

for all $p \in M$.

**Prop**: $X \underset{F}{\sim} Y$ iff for any smooth fn $f$ defined on an open subset of $N$:

$$X(f \circ F) = Yf \circ F.$$

Pf: Let $f$ be defnd on nbd of $F(p) \in N$. Then

$$X(f \circ F)(p) = X_p(f \circ F)$$

$$= dF_p(X_p) f.$$

$$\|$$

and

$$(Yf \circ F)(p) = (Yf)_{F(p)} = Y_{F(p)} f. \qquad \blacksquare$$

Ex: Let

$$F : \mathbb{R} \longrightarrow \mathbb{R}^2$$
$$t \longmapsto (\cos t, \sin t)$$

$$X = \frac{d}{dt} \in \mathrm{Vect}(\mathbb{R})$$

$$Y = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \in \mathrm{Vect}(\mathbb{R}^2)$$

Then $X \underset{F}{\sim} Y$. $\blacksquare$

<u>Prop</u>: If $F: M \to N$ is a diffeomorphism, and $X \in \mathrm{Vect}(M)$, then $\exists!\ Y \in \mathrm{Vect}(N)$ s.t. $X \overset{F}{\sim} Y$. In other words

$$F_* = \text{"}dF\text{"} : \mathrm{Vect}(M) \xrightarrow{\overset{\checkmark}{\cong}} \mathrm{Vect}(N)$$

is defined and is an isomorphism.

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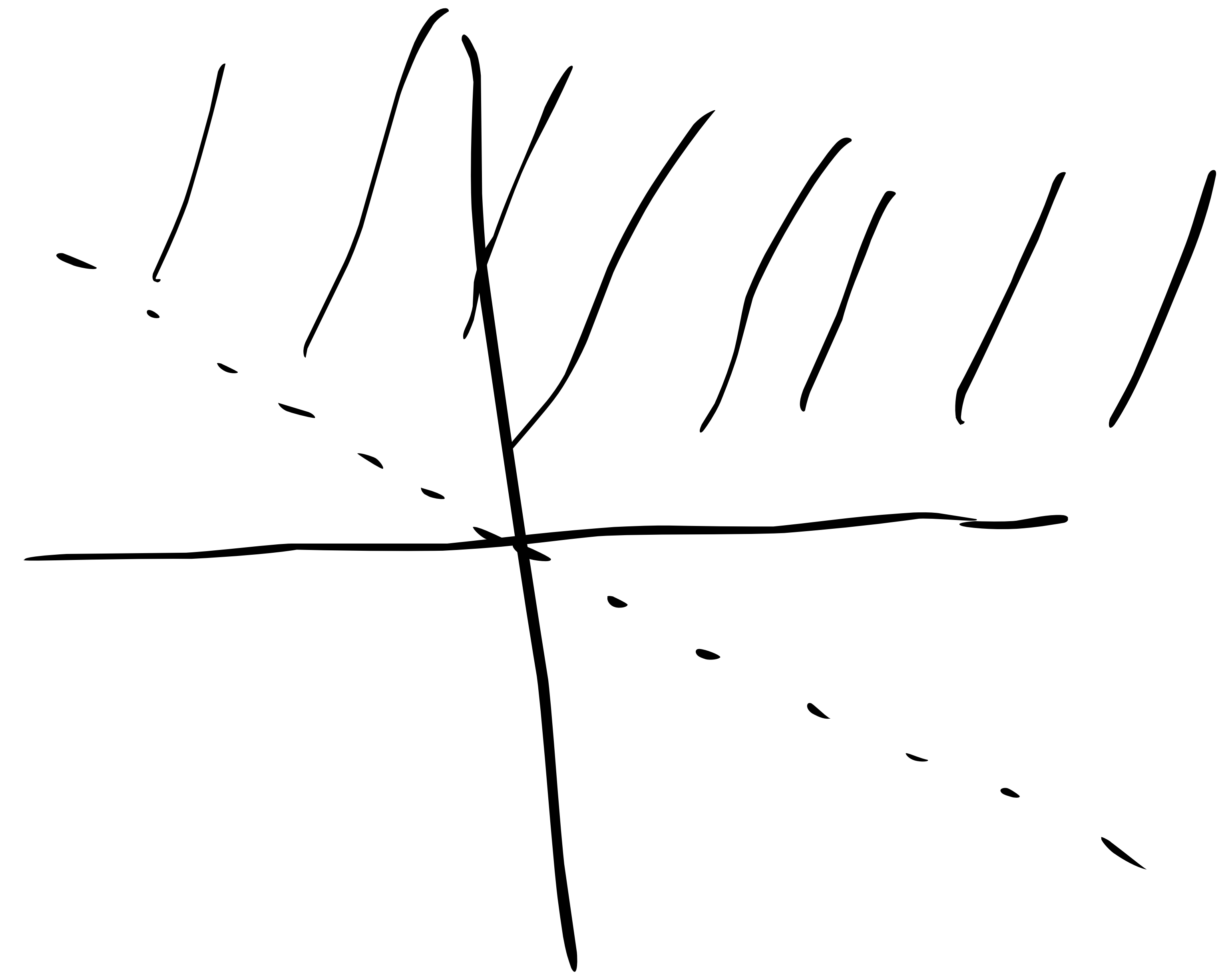
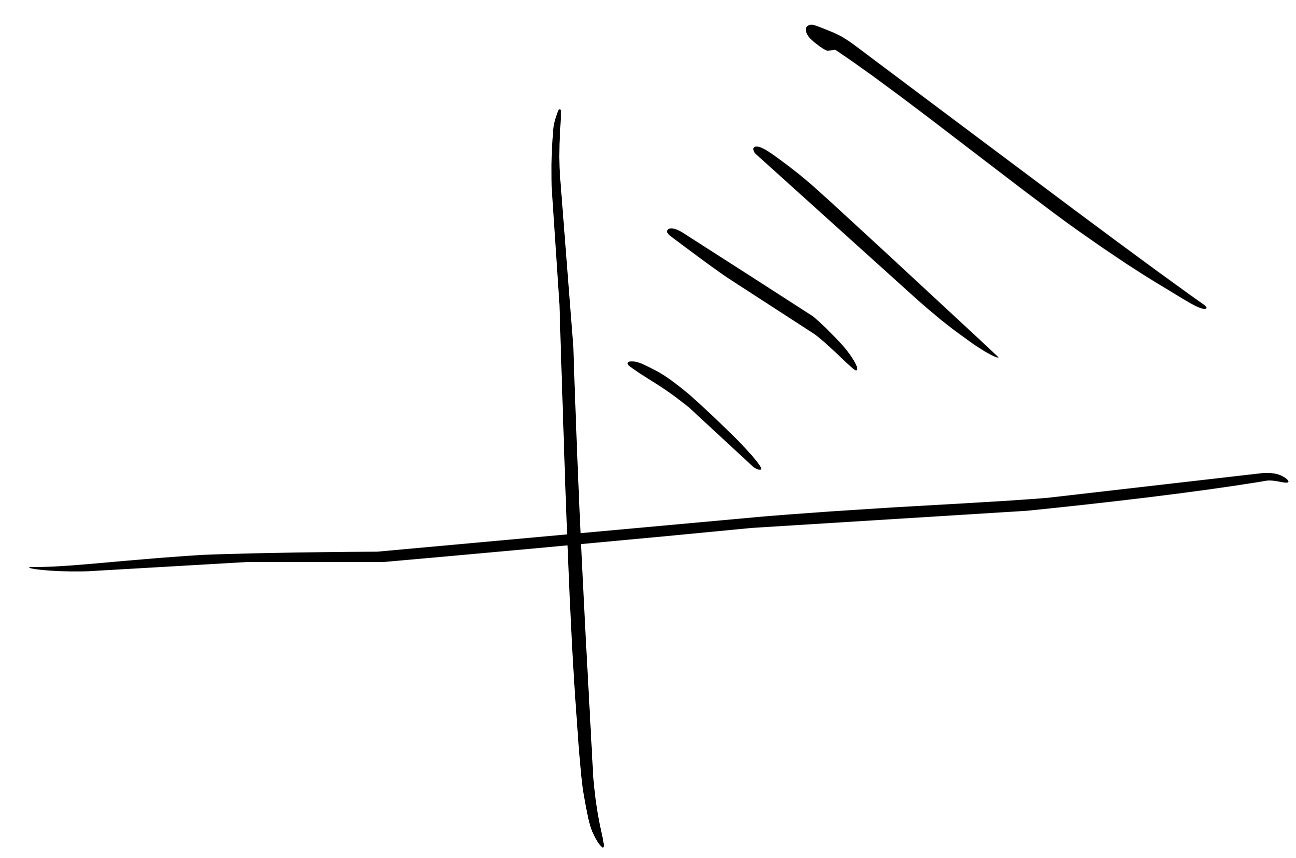
Ex:

$M =$  $\subset \mathbb{R}^2$

$N =$  $\subset \mathbb{R}^2$.

Define $F : M \to N$ by

$$F(x, y) = \left( x + y , \frac{x}{y} + 1 \right).$$

$$F^{-1}(u, v) = \left( u - \frac{u}{v} , \frac{u}{v} \right).$$

Let $X = y^2 \frac{\partial}{\partial x}$. $F_* X$ ?

$$dF_{(x,y)} = \begin{pmatrix} 1 & 1 \\ 1/y & -x/y^2 \end{pmatrix}.$$

$$dF_{F^{-1}(u,v)} = \begin{pmatrix} 1 & 1 \\ v/u & (v - v^2)/u \end{pmatrix}$$

$$X_{F^{-1}(u,v)} = \frac{u^2}{v^2} \frac{\partial}{\partial x}\Big|_{F^{-1}(u,v)}$$

$$\Rightarrow$$

$$F_* X_{(u,v)} = \frac{u^2}{v^2} \frac{\partial}{\partial u}\Big|_{(u,v)} + \frac{u}{v} \frac{\partial}{\partial v}.$$

Pf: $dF_p(X_p) = Y_{F(p)} \Rightarrow$

$$Y_q \overset{\text{def}}{=} dF_{F^{-1}(q)}\left(X_{F^{-1}(q)}\right).$$

I.e.

$$\begin{tikzcd} N \arrow[r, "F^{-1}"] \arrow[rrr, bend right, "Y"'] & M \arrow[r, "X"] & TM \arrow[r, "dF"] & TN \end{tikzcd}.$$

$\blacksquare$