# October 16

- A <u>Lie group</u> is a smooth manifold $G$ which is a group s.t.

  - multiplication $m : G \times G \to G$ is a smooth map, and

  - inversion $(-)^{-1} : G \to G$ is a smooth map.

- If $g \in G$, then the group homomorphisms

$$L_g : G \longrightarrow G, \quad h \longmapsto gh$$

$$R_g : G \longrightarrow G, \quad h \longmapsto hg$$

are smooth maps.

- A <u>Lie group homomorphism</u> is a group homomorphism which is smooth.

<u>Ex</u>: - The group of units inside a field (like $\mathbb{R}$, $\mathbb{C}$) is a Lie group.

$$\mathbb{R}^\times, \quad \mathbb{C}^\times \text{ w/ } \cdot$$

- More generally,

$$GL(n, \mathbb{R}), \quad GL(n, \mathbb{C})$$

of invertible $n \times n$ matrices are Lie groups of dimension

$$n^2, \quad 2n^2 \text{ respectively.}$$

- $S^1 \subset \mathbb{C}^\times$ is a Lie group wrt complex multiplication.

Using an angle coordinate

$$e^{i\alpha} \cdot e^{i\beta} = e^{i(\alpha+\beta)} .$$

In fact $S^1 \hookrightarrow \mathbb{C}^\times$ this embedding is a Lie group homomorphism.

<u>Thm</u>: Any Lie group homomorphism

$$\Phi : G \to H$$

is of constant rank.

<u>Pf</u>: Let $g_0 \in G$. Then for any $g \in G$:

$$\begin{aligned} \Phi(L_{g_0}(g)) &= \Phi(g_0 g) \\ &= \Phi(g_0)\Phi(g) \\ &= L_{\Phi(g_0)} \Phi(g) . \end{aligned}$$

$$\Rightarrow \quad \Phi \circ L_{g_0} = L_{\Phi(g_0)} \circ \Phi .$$

Using chain rule $\Rightarrow d(-)_e$ is

$$d\Phi_{g_0} \circ (dL_{g_0})_e = d(L_{\Phi(g_0)})_{\tilde{e}} \circ d\Phi_e$$

when $e \in G$, $\tilde{e} \in H$ are the identities.

Now, $L_{g_0}$ is a diffeomorphism, so

$\operatorname{rk} d\Phi_{g_0} = \operatorname{rk} d\Phi_e$. Since this is

true for <u>any</u> $g_0$, we are done.

$\boxtimes$

## Subgroups

A <u>Lie subgroup</u> is a subgroup $H \subset G$

s.t.

- $H$ is a Lie group.
- $i : H \longrightarrow G$ is an immersion

<u>Prop</u>: Sps $H \subset G$ is a subgroup which is also an embedded submanifold. Then $H \subset G$ is a Lie subgroup.

<u>Pf</u>: Observe

$$\begin{tikzcd}
G \times G \arrow[r] & G \\
H \times H \arrow[u, hook, "\text{embedd}"] \arrow[r] & H \arrow[u, hook, "\text{embedd}"']
\end{tikzcd}$$

$\Rightarrow$ $\underbrace{H \times H \longrightarrow}$ is smooth.

Similarly $(\cdot)^{-1} : H \to H$ is smooth $\square$

Lem: Sps $H \subset G$ is open subgroup. Then $H$ is an embedded Lie subgroup. Additionally, $H$ is closed and so

$$H = \bigsqcup_{\alpha} G_{\alpha} \quad \text{connected}.$$

Pf: Since $H$ is open it is automatically embedded. Also, the cosets

$$gH = L_g(H) \subset G$$

are open, since $L_g$ is diffeo.

$$G \setminus H = \bigcup_{\substack{g \\ gH \neq H}} gH$$

is open $\Rightarrow$ $H$ closed. ∎

- The <u>identity component</u> $G_0$ of $G$ is the connected component of $G$ containing $e \in G$.

<u>Prop</u>: $G_0 \subset G$ is a normal subgroup, and is the only open connected subgroup. Further, every other component of $G$ is diffeomorphic to $G_0$.

- By the rk theorem:

<u>Prop</u>: If $\Phi : G \to H$ is a Lie group homomorphism then

$$\ker \Phi \subset G$$
$$\parallel$$
$$\Phi^{-1}(e).$$

is an embedded subgroup.

Ex: $\cdot$ $S^1 \subset \mathbb{C}^\times$ is embedded Lie subgroup.

$\cdot$ $SL(n, \mathbb{R}) \subset GL(n, \mathbb{R})$

$$\begin{array}{c} SL(n,\mathbb{R}) \\ \parallel \\ \{A \mid \det A = 1\} \end{array}$$

is embedded Lie subgroup.

$\cdot$ Similarly $SL(n, \mathbb{C}) \subset GL(n, \mathbb{C})$.

Thm: Sps $H \subset G$ is Lie subgroup.

$H$ closed $\iff$ it is an embedded Lie subgroup.