## October 18

Recall that given

$$X \in \mathrm{Vect}(M)$$
$$f \in C^\infty(M)$$

we have defined

$$Xf \in C^\infty(M)$$

$$fX \in \mathrm{Vect}(M).$$

What about if $X, Y$ are two v.f.s. Can we "combine" them, say

$$\text{"}XY\text{"} \overset{?}{\in} \mathrm{Vect}(M).$$

Problem: $XY$ is not a vector field.

Dfn: The commutator of v.f.'s $X, Y$ is

$$[X,Y] : C^\infty(M) \longrightarrow C^\infty(M)$$

defined by

$$[X,Y]f = X(Yf) - Y(Xf)$$

Lem: $[X,Y]$ is a vector field.

Pf: $[X,Y](fg) = XY(fg) - YX(fg)$.

$$= X\big((Yf)g + fYg\big) - Y\big((Xf)g + fXg\big)$$

$$= \big(X(Yf)\big)g + Yf\,Xg + Xf\,Yg + f(X(Yg))$$

$$- \quad " \qquad "$$

$$= fXYg + gXYf - fYXg - gYXf$$

$$= f[X,Y]g + g[X,Y]f. \quad \checkmark.$$

$\boxed{//}$

Dfn: A Lie algebra (over $\mathbb{R}$) is a real vector space $\mathfrak{g}$ equipped w/ a bilinear map (called the bracket/commutator)

$$[-,-] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$$

s.t. 1) Skew symmetry $[x,y] = -[y,x]$.

2) Jacobi identity.

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0.$$

Thm: W/ the commutator above, $\mathrm{Vect}(M)$ is a Lie algebra.

Pf: Let $X, Y, Z \in \mathrm{Vect}(M)$ and $f \in C^\infty(M)$.

$$[X,[Y,Z]]f = X[Y,Z]f - [Y,Z]Xf$$

$$= XYZf - XZYf - YZXf + ZYXf$$

$[Y,[Z,X]]f =$

$$YZXf - YXZf - ZXYf + XZYf.$$

$[Z,[X,Y]]f =$

$$ZXYf - ZYXf - XYZf + YXZf.$$

$\boxed{/\!/}$

- Other examples.

• $\mathrm{Mat}_n(\mathbb{R}) = gl(n,\mathbb{R})$ is a Lie algebra w/

$$[A,B] = AB - BA.$$

Similarly $\mathrm{Mat}_n(\mathbb{C})$ is.
$= gl(n,\mathbb{C})$.

- More generally, if $V$ is any vector space then

$$\mathrm{End}\, V = \mathrm{Hom}(V, V)$$

is a Lie algebra w/

$$[\Phi, \Psi] = \Phi \circ \Psi - \Psi \circ \Phi .$$

- Back to Lie groups. Let $G$ be Lie group.

<u>**Dfn**</u>: A v.f. $X \in \mathrm{Vect}(G)$ is <u>left invariant</u> if

$$(L_g)_* X = X$$

for all $g \in G$. In other words

$$(dL_g)_h (X_h) = X_{gh}$$

for all $g, h \in G$.

**Prop**: If $X, Y$ are left invt vector fields on $G$, then so is $[X, Y]$.

This follows from the general result.

**Lemma**: Sps $F: M \to N$ is smooth and $X_i, Y_i$ are v.f.'s on $M, N$ $i=1,2$ s.t.

$$X_i \overset{F}{\sim} Y_i \qquad i=1,2.$$

Then

$$[X_1, X_2] \overset{F}{\sim} [Y_1, Y_2].$$

**Pf**: Recall $X_1 \overset{F}{\sim} Y_1 \iff$

$$X_1(f \circ F) = (Y_1 f) \circ F$$

for all $f \in C^\infty(N)$.

Now:

$$X_1(X_2(f\circ F)) = X_1(Y_2 f\circ F)$$

$$= (Y_1 Y_2 f)\circ F$$

and

$$X_2(X_1(f\circ F)) = X_2(Y_1 f\circ F)$$

$$= (Y_2 Y_1 f)\circ F.$$

$$\Rightarrow [X_1, X_2](f\circ F) = X_1(X_2(f\circ F)) - (1\leftrightarrow 2)$$

$$= Y_1 Y_2 f\circ F - Y_2 Y_1 f\circ F$$

$$= [Y_1, Y_2] f\circ F.$$ ▨

<u>Cor</u>: If $F$ is diffeomorphism then

$$F_* [X_1, X_2] = [F_* X_1, F_* X_2].$$

Pf: (of proposition)

$$(L_g)_* [X, Y] = [L_{g*} X, L_{g*} Y]$$
$$= [X, Y] \qquad \boxed{/\!/\!/}$$

Thus, the space of left invariant v.f.'s on a Lie group forms a Lie algebra. Denote it

$$\mathrm{Lie}(G).$$

In fact, it is a Lie "subalgebra" of all vfs on $G$, $\mathrm{Vect}(G)$.

Thm: $\mathrm{Lie}(G)$ is finite dimensional.