Oct 21

(Finish embedding results from last lecture.)

Dfn: A (embedded) submanifold

$$S \subset M$$

is a subset equipped w/ a smooth str. wrt. the subspace topology s.t. the inclusion

$$1|_S \overset{dfn}{=} i_S : S \hookrightarrow M$$

is a smooth embedding.

Ex: • $S^n \hookrightarrow \mathbb{R}^{n+1}$ (we prove this soon)

• $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+m}$ ...

The codimension of $S \subset M$ is

$$\operatorname{codim} S = \dim M - \dim S.$$

Prop: Open subsets of M are codim 0 submanifolds. In fact, those are all the codim 0 submanifolds.

- The dfn of a submanifold required the inclusion to be an embedding. Conversely:

Prop: Sps $F: N \to M$ is an embedding. The

$$S \overset{dfn}{=} F(N) \subset M$$

has a unique smooth str. for which

$$i_S : S \hookrightarrow M$$

endows S w/ the structure of a submanifold.

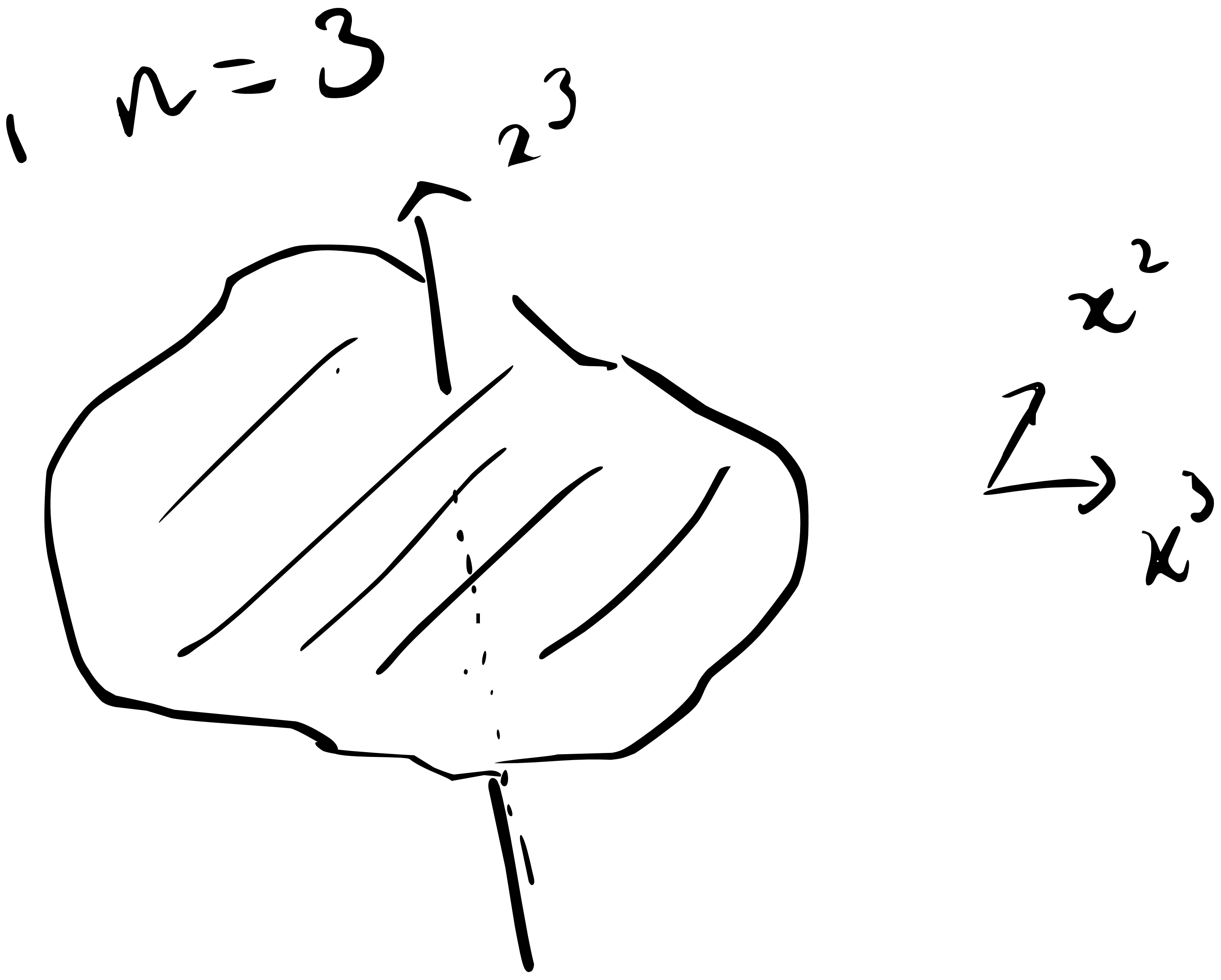
We now characterize a local form of a submanifold.

A <u>k-slice</u> of $\mathbb{R}^n$ $(k \leq n)$ is a submanifold of the form

$$U \cap \underbrace{\left\{ x = (x^i) \;\middle|\; \begin{array}{l} x^j = c^j, \quad j > k \\ \text{for constants } c^j \end{array} \right\}}_{\cong\, \mathbb{R}^k} \subset \mathbb{R}^n$$

for some open $U \subseteq \mathbb{R}^n$.

<u>Ex:</u> $k = 2$, $n = 3$



A subset $S \subset M$ satisfies the <u>k-slice condition</u> if every pt in S admits a chart (U, ϕ) containing the pt s.t.

$$\phi(U \cap S) \subset \mathbb{R}^n \text{ is a } k\text{-slice.}$$

Thm: Sps $S \subset M$ is an embedded submanifold.

Then S satisfies the k-slice condition.

Pf: Apply rank thm to $i_S : S \hookrightarrow M$.

There is a chart (U, ϕ) of S and (V, ψ) of M s.t.

$$\hat{i}_S(x^1, \dots, x^k) = (x^1, \dots, x^k, 0, \dots, 0).$$

$\parallel$

$\psi \circ i_S \circ \phi^{-1}$

Let $U_0 \subset U$ be a ball of size ε containing $p \in S$. That is
in S

$$U_0 = \phi^{-1}\Big(B_\varepsilon(\phi(p)) \Big) \subset U$$

Then by defn of subspace topology $\exists\, W \underset{\text{open}}{\subset} M$ s.t.

$$U_0 = W \cap S.$$

Now sps $V_0 \subset V$ is ball nr p in M of size $\varepsilon > 0$. Let

$$V' = V_0 \cap W.$$

S is a slice wrt the coordinate chart

$$(V', \psi|_{V'}).$$