October 23 | M = mfld.

A (*smooth*) *curve* is a (smooth) map

$$\gamma : J \longrightarrow M$$

where $J \subset \mathbb{R}$ is an interval. (usually open).

The velocity of γ at $t_0 \in J$ is

$$\gamma'(t_0) = d\gamma_{t_0}\left(\frac{d}{dt}\Big|_{t_0}\right) \in T_{\gamma(t_0)}M.$$

Note, for $f \in C^\infty(M)$ have

$$\gamma'(t_0) f = (f \circ \gamma)'(t_0).$$

Lemma: Every $v \in T_pM$ is the velocity of a smooth curve in M.

Pf: Let (U, ϕ) be chart around p, and write

$$v = v^i \frac{\partial}{\partial x^i}\bigg|_p .$$

For $\varepsilon > 0$ small enough let

$$\gamma : (-\varepsilon, \varepsilon) \longrightarrow U$$

$$t \longmapsto \phi^{-1}(tv^1, \dots, tv^n) .$$

$\blacksquare$

Prop: Sps $F : M \to N$ smooth, $v \in T_pM$. Then for any smooth curve γ passing through p, so $\gamma(0) = p$, and $\gamma'(0) = v$:

$$dF_p(v) = (F \circ \gamma)'(0) .$$

Pf: $(F \circ \gamma)'(0) = d(F \circ \gamma)_0 \left(\frac{d}{dt}\Big|_0 \right)$

$= dF_p \circ d\gamma_0 \left(\frac{d}{dt}\Big|_0 \right)$

$= dF_p \left(\gamma'(0) \right)$. $\blacksquare$

Now sps V is a vector field on M.
An integral curve for V is a smooth curve

$$\gamma : J \longrightarrow M$$

s.t.

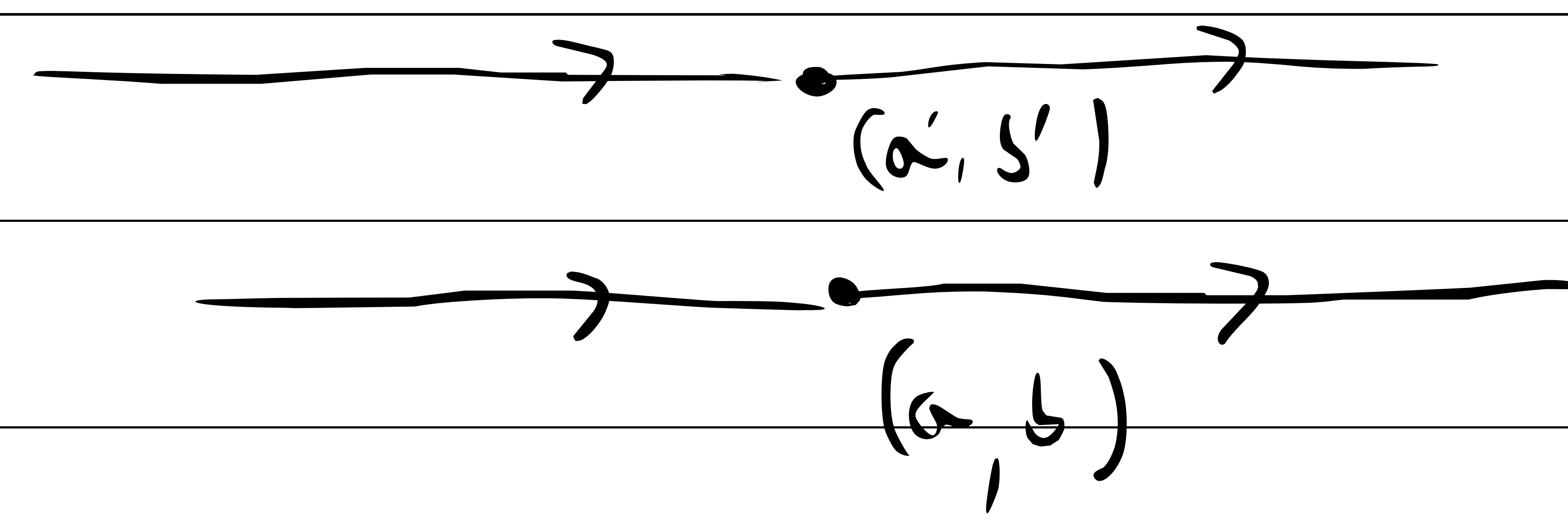
$$V_{\gamma(t)} = \gamma'(t)$$

for all $t \in J$.

Ex: • $M = \mathbb{R}^2$, $V = \frac{\partial}{\partial x}$.

Then all integral curves for V are of the form

$$\gamma(t) = (a + t, b).$$



• $M = \mathbb{R}^2$, $V = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$.

Sps $\gamma(t) = (x(t), y(t))$ is an integral curve. Then

$$\gamma'(t) = V_{\gamma(t)}$$

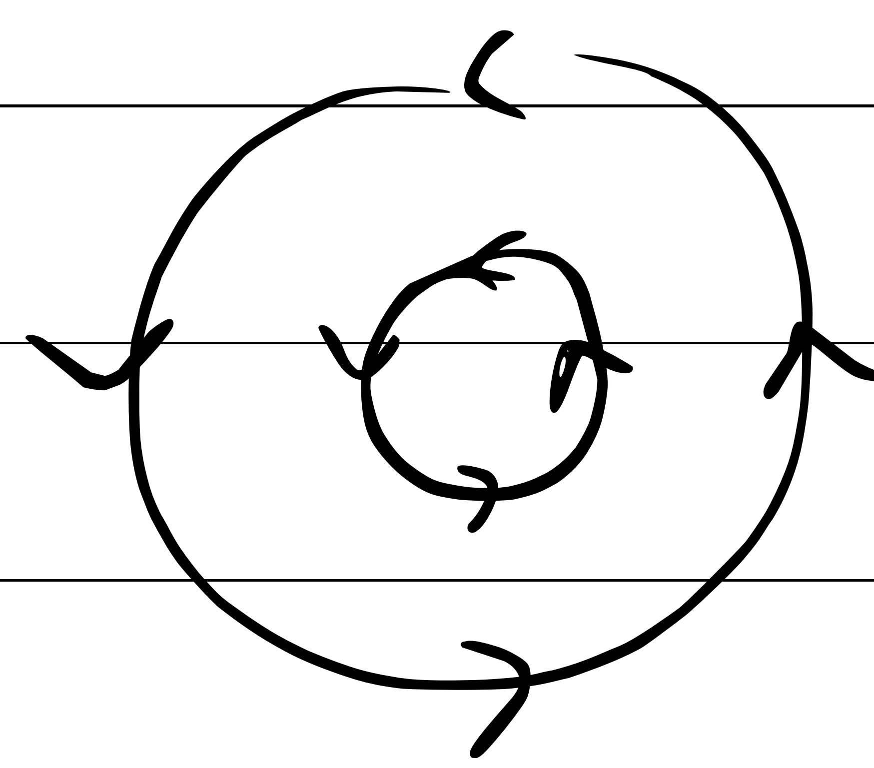
$$\iff \quad x'(t) = -y(t)$$

$$y'(t) = x(t).$$

Std system of ODE's w/ solns

$x(t) = a\cos t - b\sin t$

$y(t) = a\sin t + b\cos t.$



Existence of ODE's :

Prop: Sps V is any v.f. on M. For each $p \in M$, there is a $\varepsilon > 0$ and an integral curve

$$\gamma : (-\varepsilon, \varepsilon) \to M$$

for V.

Flows

smooth

A **global flow** on M is a smooth map

$$\theta : \mathbb{R} \times M \longrightarrow M$$
$$(t, p) \longmapsto \theta_t(p)$$

s.t. 1) $\theta_0 = \mathbb{1}_M$

2) For all $t, s \in \mathbb{R}$ have

$$\theta_t \circ \theta_s = \theta_{t+s}.$$

When we fix $p \in M$ we obtain a smooth map

$$\theta_{\cdot}(p) : \mathbb{R} \longrightarrow M$$
$$t \longmapsto \theta_t(p).$$

Prop: The assignment

$$p \in M \overset{V}{\longmapsto} \theta_{\cdot}(p)'(0)$$

defines a smooth v.f. on M. For each $p \in M$, $\theta_{\cdot}(p)$ is an integral curve for this v.f. .

Pf: Suffices to show that Vf is smooth for every $f \in C^\infty(U)$ defined on open $U \subset M$.
For such f:

$$Vf(p) = \theta_{\cdot}'(p)(0) f$$

$$= \frac{d}{dt}\left(f\left(\theta_t(p)\right)\right)\Big|_{t=0}$$

$$= \frac{\partial}{\partial t}\left(f\left(\theta(t,p)\right)\right)\Big|_{t=0}$$

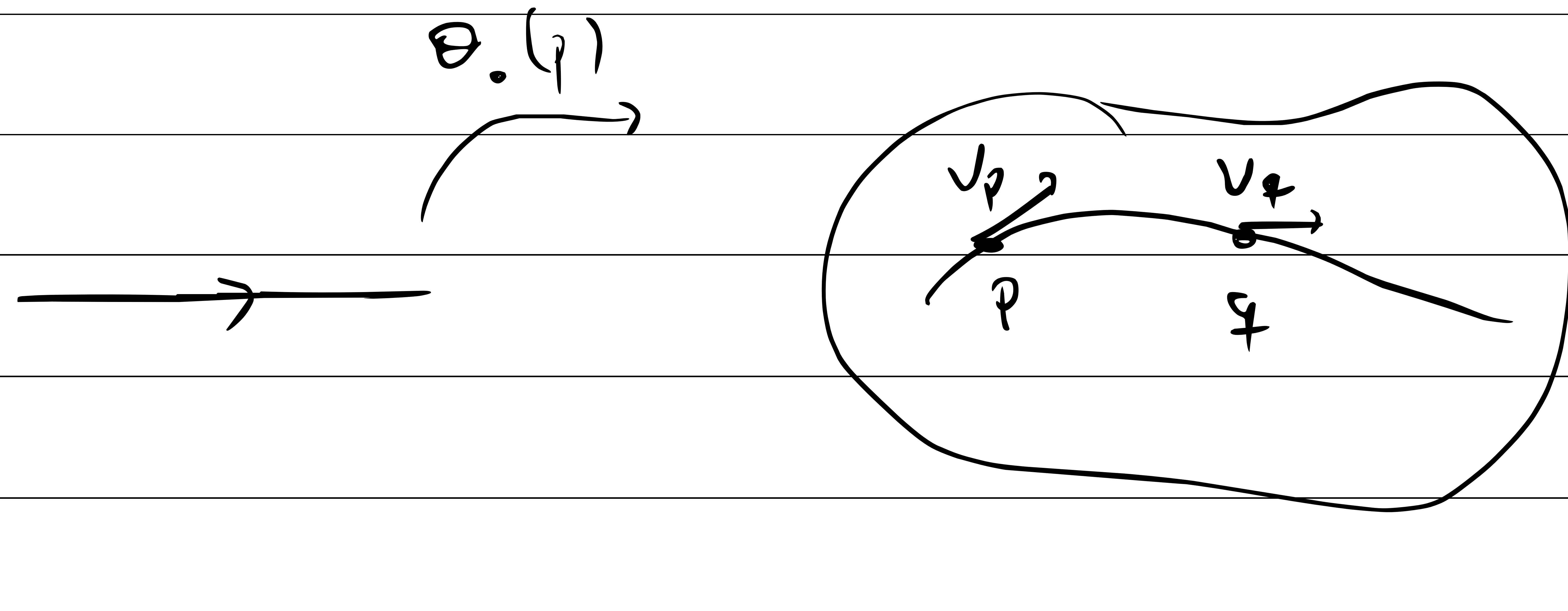
Since $(t,p) \mapsto f(\theta(t,p))$ is composition of smooth fns it is smooth and so are its partials.

Next we show $\theta_\cdot(p)$ is an integral curve of V. That is

$$\theta_\cdot(p)'(t) = V_{\theta_t(p)}$$

for all $p \in M$, $t \in \mathbb{R}$. Let t_0 be arbitrary and set $q = \theta_{t_0}(p)$. We must see

$$\theta_\cdot(p)'(t_0) = V_q .$$



By group laws:

$$\theta_t(q) = \theta_t(\theta_{t_0}(p))$$

$$= \theta_{t+t_0}(p)$$

So for f defined on a nbd of q:

$$V_q f = \theta_0(q)'(0) f$$

$$= \frac{d}{dt} f(\theta_t(q))\Big|_{t=0}$$

$$= \frac{d}{dt} f(\theta_{t+t_0}(p))\Big|_{t=0}$$

$$= \theta_{-}(p)'(t_0) f$$

$\blacksquare$