So

$$(d\theta_{-t})_{\theta_t(x)}\left(X_{\theta_t(x)}\right)$$

$$= \frac{\partial \theta^i}{\partial x^j}\left(-t, \theta_t(x)\right) X^j\left(\theta(t,x)\right) \frac{\partial}{\partial x^i}\Big|_x .$$

Because θ^i, X^j are smooth it follows that this is smooth. $\blacksquare$

October 30

Thm: $\mathcal{L}_V X = [V, X]$ for any v.f.'s V, X.

Pf: Let $R(V) = \{p \in M \mid V_p \neq 0\} \subset M$.

This is open since V is cts. We show

$$(\mathcal{L}_V X)_p = [X, Y]_p .$$

Case 1: Sps $p \in R(V)$. Then by the rank thm we can find coordinates near p s.t.

$$V = \frac{\partial}{\partial x^1} \quad \text{in local coords.}$$

Then the corresponding flow is

$$\theta_t(x) = (t + x^1, x^2, \dots, x^n).$$

$\Rightarrow$

$$(d\theta_{-t})_{\theta_t(x)} X_{\theta_t(x)} =$$

$$(d\theta_{-t})_{\theta_t(x)} \left(X^j(x^1 + t, x^2, \dots, x^n) \frac{\partial}{\partial x^j}\bigg|_{\theta_t(x)} \right)$$

$$= X^j(x^1 + t, x^2, \dots, x^n) \frac{\partial}{\partial x^j}\bigg|_x .$$

Thus

$$(L_V X)_x = \frac{d}{dt}\Big|_{t=0} X^j(x^1+t, x^2, \dots, x^n) \frac{\partial}{\partial x^j}\Big|_x$$

$$= \frac{\partial X^j}{\partial x^1}(x^1, \dots, x^n) \frac{\partial}{\partial x^j}\Big|_x .$$

$$= \left[\frac{\partial}{\partial x^1}, X = X^j \frac{\partial}{\partial x^j}\right]_x .$$

<u>Case 2</u> : $p \in \overline{R(V)}$ = closure of $R(V)$.

This case follows by continuity.

<u>Case 3</u> : $p \in M \setminus \overline{R(V)}$. Then $V \equiv 0$ in some nbd of p. So $\theta_t \equiv \mathbb{1}$ in this same nbd, and

$$(d\theta_{-t})_{\theta_t(p)} X_{\theta_t(p)} = X_p \Rightarrow L_V X \equiv 0 .$$

$\blacksquare$

October 30

Locally, we know TM looks like

$$M \times \mathbb{R}^n$$

In particular, for each $p \in M$, T_pM has the structure of a vector space ($\mathbb{R}^n$).

A vector bundle is a generalization of this. Roughly, it is a family of vector spaces parametrized by points in M which locally looks like $M \times V$.

<u>Dfn</u>: A <u>vector bundle</u> of rank k on M is a top'l space E together w/ a continuous map

$$\pi : E \longrightarrow M$$

s.t.

1) For $p \in M$, $\pi^{-1}(p) = E_p$ has the str. of a $\mathbb{R}$-vector space.

2) For $p \in M$, $\exists$ $U \ni p$ nbd and a homeomorphism

$$\psi : \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^k$$

s.t. $\cdot$ $p_1 \circ \psi = \psi$

$$\begin{tikzcd} \pi^{-1}U \arrow[r, "\psi"] \arrow[dr, "\pi"'] & U \times \mathbb{R}^k \arrow[d, "p_1"] \\ & U \end{tikzcd}$$

$\cdot$ for each $q \in U$ the map

$$\begin{tikzcd} E_q \arrow[r, dashed, "\cong"] \arrow[d, hook] & \mathbb{R}^k \arrow[d, hook, "\{q\} \times \mathbb{1}"] \\ \pi^{-1}(U) \arrow[r, "\psi"] & U \times \mathbb{R}^k \end{tikzcd}$$

is a vector space isomorphism.

* We unpack this last condition. Since $p_1 \circ \psi = \psi$, we see that the

restriction $\psi\big|_{E_q}$ defines a map $E_q \to \{q\} \times \mathbb{R}^k$
$$\shortparallel$$
$$\mathbb{R}^k.$$

- If M, E are smooth manifolds, π is a smooth map, and all ψ's are smooth then we say this is a <u>smooth</u> vector bundle.

- A rank 1 vector bundle is called a <u>line bundle</u>.

- E is called the "total space" and π the "projection".

- For any k, there is the trivial rank k vector bundle

$$\begin{array}{ccc} E & = & M \times \mathbb{R}^k. \\ \pi \big\downarrow & & \big\downarrow p_1 . \\ M & = & M \end{array}$$