

October 4

If  $\Phi: M \rightarrow N$  is a smooth map and  $c \in N$ , then

$$\Phi^{-1}(c) \subset M$$

is called a level set of  $\Phi$ .

Ex:  $\Phi: \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $\Phi(x, y) = x^2 - y$ .

Then  $\Phi^{-1}(0) = \text{graph of } y = x^2$ .

$\leadsto$  This is a smooth submanifold of  $M$ .

Not all level sets are submanifolds.

Ex:  $\Phi: \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $\Phi(x, y) = x^2 - y^2$ . Then

$\Phi^{-1}(0)$  is not a submanifold of  $\mathbb{R}^2$ .

Thm: Spcs  $\Phi$  is of constant rank  $r$ .

Then for all  $c \in N$  the level set

$$\Phi^{-1}(c) \subset M$$

is a smooth submanifold of

dimension  $n$ .

Pf:  $m = \dim M$ ,  $n = \dim N$ . By rank

theorem  $\exists$  smooth charts  $(U, \phi)$  centered

at  $p \in \Phi^{-1}(c)$  and  $(V, \psi)$  centered

at  $c \in N$  s.t.

$$\hat{\Phi}(x^1, \dots, x^m) = (x^1, \dots, x^r, 0, \dots, 0)$$

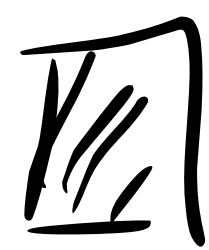
$\Rightarrow S \cap U$  is the slice

$$\left\{ (x^1, \dots, x^r, x^{r+1}, \dots, x^m) \mid x^1 = \dots = x^r = 0 \right\}$$

$\cap$   
 $U$

So  $S$  satisfies the  $k$ -slice condition

w/  $k = m - r = \text{codim } S$ .



Cor: If  $\Phi: M \rightarrow N$  is a smooth submersion

then each level set is a submanifold

of  $\text{codim} = \dim N$ .

These statements are global on the target,

but can be strengthened if we are only

concerned about a specific level set.

Dfn: • A pt  $p \in M$  is a regular point

of  $\Phi: M \rightarrow N$  if  $d\Phi_p: T_p M \rightarrow T_{\Phi(p)} N$

is surjective. Otherwise  $p \in M$  is a

critical point.

• A pt  $c \in N$  is a regular value of  $\Phi$  if every  $q \in \Phi^{-1}(c)$  is a regular pt.

Otherwise,  $c \in N$  is a critical value.

•  $\Phi^{-1}(c)$  is a regular level set if  $c \in N$  is a regular value.

Prop: Every regular level set of  $\Phi$

is a submanifold w/  $\dim = \dim N$ .

Pf:  $U = \left\{ q \in M \mid \text{rank } d\Phi_q = \dim N \right\}$

is an open subset of  $M$  (we proved

this a few lectures ago, basically

follows from the fact that matrices of full rank is an open subset of

all matrices). By assumption

$$\Phi^{-1}(c) \subset U.$$

$\Rightarrow \Phi|_U : U \rightarrow N$  is a submersion

and by last corollary  $\Phi^{-1}(c) \subset U$  is a submanifold.  $\square$

Ex: Let  $\Phi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  be

$$\Phi(x) = |x|^2.$$

The

$$d\Phi_x(v) = 2 \sum_i v_i x_i.$$

$\Rightarrow d\Phi_x$  is surjective when  $x \neq 0$ .

$\Rightarrow \Phi^{-1}(c \neq 0) \subset \mathbb{R}^{n+1}$  is submanifold.  
"2  $S^n$

## Tangent space to a submanifold.

Let  $i : S \hookrightarrow M$  submanifold.

The  $di_p : T_p S \rightarrow T_p M$  is injective

Explicitly, in terms of derivations if

$$\tilde{v} = di_p(v) \in T_p M$$

then for  $f \in C^\infty(M)$ :

$$\tilde{v}(f) = v(f \circ i) = v(f|_S)$$

Restriction of smooth map to submanifold is smooth.

Prop: (See HW) As a subspace of  $T_p M$ :

$$T_p S = \left\{ v \in T_p M \mid vf = 0 \text{ when } f|_S = 0 \right\}$$

Pf: (sketch)  $\text{sps } \nu \in T_p S \subset T_p M$ .

So  $\nu = di_p(\omega)$  for some  $\omega \in T_p S$ .

If  $f \in C^\infty(M)$  satisfies  $f|_S \equiv 0$  then

$$\nu f = \omega(f \circ i) = \omega(f|_S) = 0.$$

Conversely  $\text{sps } \nu f = 0$  whenever  $f|_S \equiv 0$ .

We need to show  $\exists \omega \in T_p S$  s.t.

$$\nu = di_p(\omega)$$

Let  $(x^1, \dots, x^n)$  be slice coordinates for  $S$  in some nbd  $U \ni p$ . Then

$$U \cap S = \left\{ x^{k+1} = \dots = x^n = 0 \right\}$$

and

$$i|_U(x^1, \dots, x^k) = (x^1, \dots, x^k, 0, \dots, 0)$$

.....

$\square$