October 4

If $\Phi : M \longrightarrow N$ is a smooth map and $c \in N$, then

$$\Phi^{-1}(c) \subset M$$

is called a <u>level set</u> of Φ.

<u>Ex</u>: $\Phi : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $\Phi(x, y) = x^2 - y$.

Then $\Phi^{-1}(0) =$ graph of $y = x^2$.

$\rightsquigarrow$ This is a smooth submanifold of M.

Not all level sets are submanifolds.

<u>Ex</u>: $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}$, $\Phi(x, y) = x^2 - y^2$. Then

$\Phi^{-1}(0)$ is <u>not</u> a submanifold of $\mathbb{R}^2$.

Thm: Sps Φ is of constant rank r. Then for all $c \in N$ the level set

$$\Phi^{-1}(c) \subset M$$

is a smooth submanifold of codimension r.

Pf: $m = \dim M$, $n = \dim N$. By rank theorem $\exists$ smooth charts (U, ϕ) centered at $p \in \Phi^{-1}(c)$ and (V, ψ) centered at $c \in N$ s.t.

$$\hat{\Phi}(x^1, \dots, x^m) = (x^1, \dots, x^r, 0 \dots 0)$$

$\Rightarrow$ $S \cap U$ is the slice

$$\{(x^1, \dots, x^r, x^{r+1}, \dots, x^m) \mid x^1 = \dots = x^r = 0\}$$
$$\cap$$
$$U$$

So S satisfies the k-slice condition w/ $k = m - r = \operatorname{codim} S$. $\blacksquare$

Cor: If $\Phi : M \to N$ is a smooth submersion then each level set is a submanifold of codim $= \dim N$.

These statements are global on the target, but can be strengthened if we are only concerned about a specific level set.

Dfn: • A pt $p \in M$ is a <u>regular point</u> of $\Phi : M \to N$ if $d\Phi_p : T_pM \to T_{\Phi(p)}N$ is surjective. Otherwise $p \in M$ is a <u>critical point</u>.

- A pt $c \in N$ is a <u>regular value</u> of Φ if every $p \in \Phi^{-1}(c)$ is a regular pt. Otherwise, $c \in N$ is a <u>critical value</u>.
- $\Phi^{-1}(c)$ is a <u>regular level set</u> if $c \in N$ is a regular value.

<u>Prop</u>: Every regular level set of Φ is a submanifold w/ codim $=$ dim N.

<u>Pf</u>: $U = \{ p \in M \mid \text{rank } d\Phi_p = \dim N \}$ is an open subset of M (we proved this a few lectures ago, basically follows from the fact that matrices of full rank is an open subset of

all matrices). By assumption

$$\Phi^{-1}(c) \subset \mathcal{U}.$$

$\Rightarrow$ $\Phi|_{\mathcal{U}} : \mathcal{U} \longrightarrow N$ is a submersion

and by last corollary $\Phi^{-1}(c) \subset \mathcal{U}$ is a submanifold. $\boxtimes$

<u>Ex</u>: Let $\Phi : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ be

$$\Phi(x) = |x|^2.$$

Then

$$d\Phi_x(v) = 2 v_i x^i$$

$\Rightarrow$ $d\Phi_x$ is surjective when $x \neq 0$.

$\Rightarrow$ $\underbrace{\Phi^{-1}(c \neq 0)}_{\cong S^n} \subset \mathbb{R}^{n+1}$ is submanifold.

Tangent space to a submanifold.

Let $i : S \hookrightarrow M$ submanifold.

Then $di_p : T_p S \to T_p M$ is injective

Explicitly, in terms of derivations if

$$\tilde{v} = di_p(v) \in T_p M$$

then for $f \in C^\infty(M)$:

$$\tilde{v}(f) = v(f \circ i) = v\left(f\big|_S\right)$$

Restriction of smooth map to submanifold is smooth.

<u>Prop</u>: (See HW) As a subspace of $T_p M$:

$$T_p S = \left\{ v \in T_p M \;\middle|\; vf = 0 \text{ when } f\big|_S = 0 \right\}.$$

Pf: (sketch) Sps $v \in T_pS \subset T_pM$.

So $v = di_p(w)$ for some $w \in T_pS$.

If $f \in C^\infty(M)$ satisfies $f|_S \equiv 0$ then

$$vf = w(f \circ i) = w(f|_S) = 0.$$

Conversely sps $vf = 0$ whenever $f|_S \equiv 0$.
We need to show $\exists\, w \in T_pS$ s.t.

$$v = di_p(w).$$

Let $(x^1, \dots, x^n)$ be slice coordinates for S in some nbd $U \ni p$. Then

$$U \cap S = \{ x^{k+1} = \dots = x^n = 0 \}.$$

and

$$i|_U(x^1, \dots, x^k) = (x^1, \dots, x^k, 0, \dots, 0).$$

$\boxtimes$

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