September 11 We have introduced the

total derivative DF and the Jacobian

JF made up of all partial derivatives.

Prop: Sps $F : \mathbb{R}^n \supset U \to \mathbb{R}^m$ is d'ble at $a \in U$.
Then

$$DF(a) = JF(a)$$

as linear maps $\mathbb{R}^n \longrightarrow \mathbb{R}^m$.

Pf: Let $J = DF(a)$, and

$$R(v) = F(a+v) - F(a) - Jv$$

when v is "small" so that $a+v \in U$.

F d'ble $\Rightarrow$ $\dfrac{R(v)}{|v|} \xrightarrow{v \to 0} 0$.

$$\frac{\partial F^i}{\partial x^j}(a) = \lim_{t \to 0} \frac{F^i(a + te_j) - F^i(a)}{t}$$

$$= \lim_{t \to 0} \frac{J^i_j t + R^i(te_j)}{t}$$

$$= J^i_j + \lim_{t \to 0} \frac{R^i(te_j)}{t}$$

$$= J^i_j. \qquad \blacksquare$$

Like in calculus, we can consider derivatives as functions of the point that we are evaluating at:

$$DF : U \longrightarrow \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$$

$$a \in U \longmapsto DF(a).$$

If
$$F : U \longrightarrow \mathbb{R}^m$$
is such that all partials exist at $a \in U$, we say F is d'ble at a. If d'ble at $\underline{\underline{\text{all}}}$ $a \in U$

and $$\frac{\partial F^i}{\partial x^j} : U \longrightarrow \mathbb{R} \text{ is cts } \forall\, i, j$$

then we say F is continuously differentiable.

$$C^1(U, \mathbb{R}^m) = \left\{ \begin{array}{c} \text{space of all ctsly d'ble} \\ F : U \longrightarrow \mathbb{R}^m. \end{array} \right\}$$

Similarly if
$$\frac{\partial^2 F^i}{\partial x^j \partial x^k} : U \longrightarrow \mathbb{R}$$

exist and is cts $\forall\, i, j, k$ we say that

F is C^2. Iterate this to obtain the space of C^k functions.

Def^n: • We say F is smooth if it is of class C^k for all $k \geq 0$.

• A diffeomorphism is a map

$$\begin{array}{ccc} F : \mathcal{U} & \longrightarrow & V \\ \cap & & \cap \\ \mathbb{R}^n & & \mathbb{R}^m \end{array}$$

which is [1] smooth, [2] bijective, and [3] F^{-1} is smooth.

Prop: Sps $F : \mathcal{U} \to V \subset \mathbb{R}^m$ is smooth. Then F is diffeomorphism $\Rightarrow DF(a) = JF(a)$ is a linear isomorphism for all $a \in \mathcal{U}$.

Pf: Let F^{-1} be inverse. Note that if $\mathbb{1}_{\mathcal{U}} : \mathcal{U} \to \mathcal{U}$ then

$$(D\mathbb{1}_{\mathcal{U}})(a) = \mathbb{1}_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n.$$

By chain rule

$$\mathbb{1} = D(F^{-1} \circ F)(a)$$

$$= (DF^{-1})(F(a)) \circ DF(a)$$

$\Rightarrow DF(a)$ invertible w/

$$DF(a)^{-1} = (DF^{-1})(F(a)).$$

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Let M^n be top^l mfld. Recall that charts $(U, \phi), (V, \psi)$ are <u>smoothly compatible</u> if

$$\begin{tikzcd} \phi(U \cap V) \arrow[r, "\phi^{-1}"] \arrow[rr, bend right] & U \cap V \arrow[r, "\psi"] & \psi(U \cap V) \end{tikzcd}$$

$$\underset{\mathbb{R}^n}{\cap} \qquad\qquad\qquad \underset{\mathbb{R}^n}{\cap}$$

is smooth.

- An <u>atlas</u> on M is a collection

$$\mathcal{A} = \{ (U_\alpha, \phi_\alpha) \}_{\alpha \in I}$$

of charts s.t. $M = \bigcup_\alpha U_\alpha$.

- A <u>smooth atlas</u> is an atlas s.t. all charts w/ $U_\alpha \cap U_\beta \neq \emptyset$ are smoothly compatible.

- A <u>smooth structure</u> on top^l mfld M is a maximal smooth atlas.
  [Meaning that if U_α is chart and it is smoothly cptble w/ $V \subset M$ then $V = U_\beta$ for some β.]

- A <u>smooth manifold</u> is a top^l manifold w/ a smooth structure.

Ex: • $\mathbb{R}^n$ has standard smooth str. $\{(\mathbb{R}^n, \mathbb{1})\}$.

• $\mathbb{R}$ has distinct smooth str. $\{(\mathbb{R}, f : \mathbb{R} \to \mathbb{R},\ x \mapsto x^3)\}$.

This is not compatible w/ standard smooth str.
since $f^{-1}(x) = x^{1/3}$ is not smooth.