September 13

We continue with examples of smooth manifolds.

- Graphs of smooth fn's. Let

$$F : \underset{\mathbb{R}^n}{\overset{}{U}} \longrightarrow \mathbb{R}^m \quad \text{be smooth.}$$

$U \subset \mathbb{R}^n$

Then
$$\operatorname{graph}(F) = \{ (x, F(x)) \mid x \in U \}$$
$$\subset U \times \mathbb{R}^m$$

has the str. of a smooth mfld.

Pf: Let
$$\phi : \operatorname{graph}(F) \longrightarrow U$$
$$(x,y) \longmapsto x \; .$$

ϕ is cts being the restriction of the projection. It's a homeomorphism w/ inverse

$$\phi^{-1}(x) = (x, F(x)).$$

$\Rightarrow$ graph(F) is a top^l mfld. To get a smooth structure we simply declare that

$$(\text{graph}(F), \phi) \text{ is smooth.} \qquad \boxed{///}$$

- Spheres. For $i = 1, \dots, n+1$ let

$$U_i^{\pm} = \left\{ (x^1, \dots, x^{n+1}) \;\middle|\; x^i \gtrless 0 \right\}.$$

Define $f : \mathbb{B}^n \longrightarrow \mathbb{R}$ by

$$\mathbb{B}^n = \{u \mid |u| < 1\} \qquad f(u) = \sqrt{1 - |u|^2}.$$

Then

$$U_i^{\pm} = \text{graph}\left(x^i = \pm f(x^1, \dots, \hat{x}^i, \dots x^{n+1}) \right).$$

Thus each $\mathcal{U}_i^{\pm} \cap S^n$ is locally Euclidean.
charts are

$$\phi_i^{\pm} : \mathcal{U}_i^{\pm} \cap S^n \longrightarrow \mathbb{B}^n$$
$$x^1, \dots x^{n+1} \longmapsto (x^1 \dots, \hat{x}^i, \dots x^{n+1}).$$

Transition maps: for $i < j$

$$\phi_i^{\pm} \circ (\phi_j^{\pm})^{-1} (u^1 \dots u^n)$$
$$= \Big(\underbrace{\underbrace{u^1 \dots \hat{u}^i}_{i} \dots \pm \sqrt{1-|u|^2}}_{j}, \dots u^n \Big)$$

For $i = j$

$$\phi_i^+ \circ (\phi_i^-)^{-1} = \phi_i^- \circ (\phi_i^+)^{-1} = \mathbb{1}_{\mathbb{B}^n}.$$

$\rightsquigarrow \{(\mathcal{U}_i^{\pm}, \phi_i^{\pm})\}$ defines a smooth atlas.

(Smooth structures are too rigid to classify smooth manifolds. We will see notion of equivalence shortly...)

- Any vector space is a smooth mfld.

Pf: Choose a basis $\{e_i\}$ of V. Define

$$\phi : \mathbb{R}^n \longrightarrow V$$

$$\begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \longmapsto e_i$$

Then (V, ϕ^{-1}) is a chart. Sps $\{\tilde{e}_i\}$ is another basis w/ chart $(V, \tilde{\phi}^{-1})$.

The transition fn $\phi^{-1} \circ \tilde{\phi}$ is certainly smooth. The collection of all such charts defines a smooth structure.

2^n

- Sps M is smooth manifold and $U \subset M$ is open, then U has an induced smooth structure obtained by restricting the smooth atlas on M.

- **Prop:** The space of invertible $n \times n$ matrices has the natural structure of a smooth manifold.

  Pf:

$$GL_n \subset M_{n\times n} \cong \mathbb{R}^{n^2}$$

$$\begin{aligned} GL_n &= \{A \mid \det A \neq 0\} \\ &= \det^{-1}(\mathbb{R}^\times). \end{aligned}$$

Since $\det : M_{n\times n} \to \mathbb{R}$ is cts it follows that $GL_n \subset M_{n\times n}$ is an open subset. $\blacksquare$

To construct more elaborate examples of smooth manifolds we will use the following 'reconstruction' theorem.

It is similar in sprit to constructing a topology from a basis

<u>Thm</u>: Let M be a set and spse

$$\{(U_\alpha \subset M, \phi_\alpha : U_\alpha \to \mathbb{R}^n)\}$$

is a collection satisfying:

1) $\phi_\alpha : U_\alpha \to \phi(U_\alpha)$ is bijection. $\forall \alpha$.

2) $\phi_\alpha(U_\alpha \cap U_\beta)$, $\phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$

are open $\forall \alpha, \beta$.

3) $\phi_\alpha(U_\alpha \cap U_\beta) \xrightarrow{\phi_\alpha^{-1}} U_\alpha \cap U_\beta \xrightarrow{\phi_\beta} \phi_\beta(U_\alpha \cap U_\beta)$

is smooth $\forall \alpha, \beta$.

4) $\bigcup_\alpha U_\alpha = M$ is a countable union which covers M.

5) If $p \neq q \in M$ then $\exists\ \alpha, \beta$ s.t. $p \in U_\alpha,\ q \in U_\beta$.

Then M has a smooth str. s.t. (U_α, ϕ_α) is a smooth chart for all α.

Pf: We fix a topology on M by defining a basis.

$$\left\{ \phi_\alpha^{-1}(V) \;\middle|\; V \subset_{\text{open}} \mathbb{R}^n \right\}.$$

To see its a basis: if $p \in \phi_\alpha^{-1}(V) \cap \phi_\beta^{-1}(V')$ then $\exists\ W \subset_{\text{open}} \mathbb{R}^n$ s.t.

$$\phi_\gamma^{-1}(W) \subset \phi_\alpha^{-1}(V) \cap \phi_\beta^{-1}(V')$$

for some γ. Suffices to show $\mathcal{B}$ is a basis set. Observe

$$(\phi_\beta \circ \phi_\alpha^{-1})^{-1}(V') \subset \phi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$$

is open since (3) says $\phi_\beta \circ \phi_\alpha^{-1}$ is smooth

is open by (2). Then

$$\phi_\alpha^{-1}(V) \cap \phi_\beta^{-1}(V') = \phi_\alpha^{-1}\left(V \cap (\phi_\beta \circ \phi_\alpha^{-1})^{-1}(V')\right)$$

is open. By (1), (4), (5) we see that M is a top^l manifold.

Then (3) $\Rightarrow$ $\{(U_\alpha, \phi_\alpha)\}$ is a smooth atlas.

To get smooth structure just take the maximal atlas containing this one. $\blacksquare$

Next we will use the reconstruction lemma to construct a non-trivial example. We will not discuss this in class.

- Let V be a vector space, define

$$Gr_k(V) = \left\{ \begin{array}{l} W \subset V \text{ linear subspace} \\ \dim W = k \end{array} \right\}.$$

called the Grassmannian manifold of k-planes in V. We will construct a smooth str.

Let $P, Q \subset V$ be complimentary

$$V = P \oplus Q, \quad \begin{array}{l} \dim P = k \\ \dim Q = n-k. \end{array}$$

Let

$$\mathcal{U}_Q = \left\{ S \subset V \;\middle|\; \begin{array}{l} \dim S = k \\ S \cap Q = \phi \end{array} \right\}.$$

$$\subset Gr_k(V)$$

and define

$$\Gamma : \mathrm{Hom}(P, Q) \longrightarrow \mathcal{U}_Q$$

by

$$\Gamma(L) = \left\{ v + Lv \;\middle|\; v \in P \right\}.$$

<u>Lem</u>: Γ is an isomorphism.

<u>Pf</u>: Sps $S \subset V$ is k-dim$^{\ell}$ and $S \cap Q = \phi$.

$$\begin{tikzcd} & V \arrow[dl, "\pi_P"'] \arrow[dr, "\pi_Q"] & \\ P & & Q \end{tikzcd}$$

By assumption $\pi_P\big|_S : S \longrightarrow P$ is an iso.

Define

$$L = \pi_Q \circ \left(\pi_P\big|_S\right)^{-1} : P \longrightarrow Q.$$

Then L is a linear map s.t.

$$S = \Gamma(L). \qquad ∎$$

So, $\phi_Q = L^{-1} : \mathcal{U}_Q \longrightarrow \mathrm{Hom}(P, Q)$

$$\mathrm{Hom}(P,Q) \cong \mathbb{R}^{k(n-k)}$$

is a chart for the open set $\mathcal{U}_Q$. Item (1) of the reconstruction lemma automatically holds.

Now sps P', Q' is another pair of complimentary subspaces of V. Have

$$\phi_Q(\mathcal{U}_Q \cap \mathcal{U}_{Q'}) \subset \mathrm{Hom}(P, Q)$$

$$= \left\{ L : P \to Q \;\middle|\; \Gamma(L) \cap Q' = \phi \right\}.$$

<u>Lemma</u>: This subset is open, hence (2) of reconstruction holds.

Leave this an exercise.

Next we show transition maps are smooth.

$$\phi_{Q'} \circ \phi_Q^{-1} : \phi_Q(\mathcal{U}_Q \cap \mathcal{U}_{Q'}) \to \phi_{Q'}(\mathcal{U}_Q \cap \mathcal{U}_{Q'}).$$

$$\begin{matrix} \phi_Q(\mathcal{U}_Q \cap \mathcal{U}_{Q'}) & \to & \phi_{Q'}(\mathcal{U}_Q \cap \mathcal{U}_{Q'}) \\ \Psi & & \Psi \\ L & \longmapsto & L' \end{matrix}$$

Let $I_L : P \xrightarrow{\cong} S = \Gamma(L)$ be

$$I_L(v) = v + Lv.$$

Have $L' = \pi_{Q'} \circ (\pi_{P'}|_S)^{-1}$

$$= \pi_{Q'} \circ I_L \circ I_L^{-1} \circ (\pi_{P'}|_S)^{-1}$$

$$= (\pi_{Q'} \circ I_L) \circ (\pi_{P'} \circ I_L)^{-1}$$

We need to see that L' depends smoothly on L. - it suffices to see that in a basis, the matrix entries of L' depend smoothly on those of L.

Let

$A = \pi_{P'}|_P$, $B = \pi_{Q'}|_P$, $C = \pi_{P'}|_Q$, $D = \pi_{Q'}|_Q$.

Then

$$\pi_{P'} \circ I_L = A + C \circ L$$

$$\pi_{Q'} \circ I_L = B + D \circ L .$$

$$\Rightarrow L' = (B + D \circ L) \circ (A + C \circ L)^{-1} .$$

By Cramers formula, the matrix entries of $(A + CL)^{-1}$ depend smoothly on those of L. This proves (3) of reconstruction.

I'll leave the rest of the verifications as an exercise (or consult Lee).

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Now we continue with the lecture.

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→ Next we move on to smooth fns.

Sps M is a top^l manifold, and let

$$f : M \to \mathbb{R}^m$$

be a function defined on all of M.

We say f is <u>smooth</u> if ~~for~~ all $p \in M$ there exists a chart (U, ϕ) near $p \in M$ s.t.

$$f \circ \phi^{-1}\big|_{\phi(U)} : \begin{array}{c} \phi(U) \\ \cap \\ \mathbb{R}^n \end{array} \xrightarrow{\phi} U \xrightarrow{f} \mathbb{R}^m$$

is smooth.

<u>Lemma</u>: Spo $f : M \to \mathbb{R}^m$ is smooth and (V, ψ) is any chart for M. Then $f \circ \psi^{-1}$ is smooth.

<u>Pf</u>: Let $p \in \psi(V)$, and let (U, ϕ) be a chart near $\psi^{-1}(p)$ s.t.

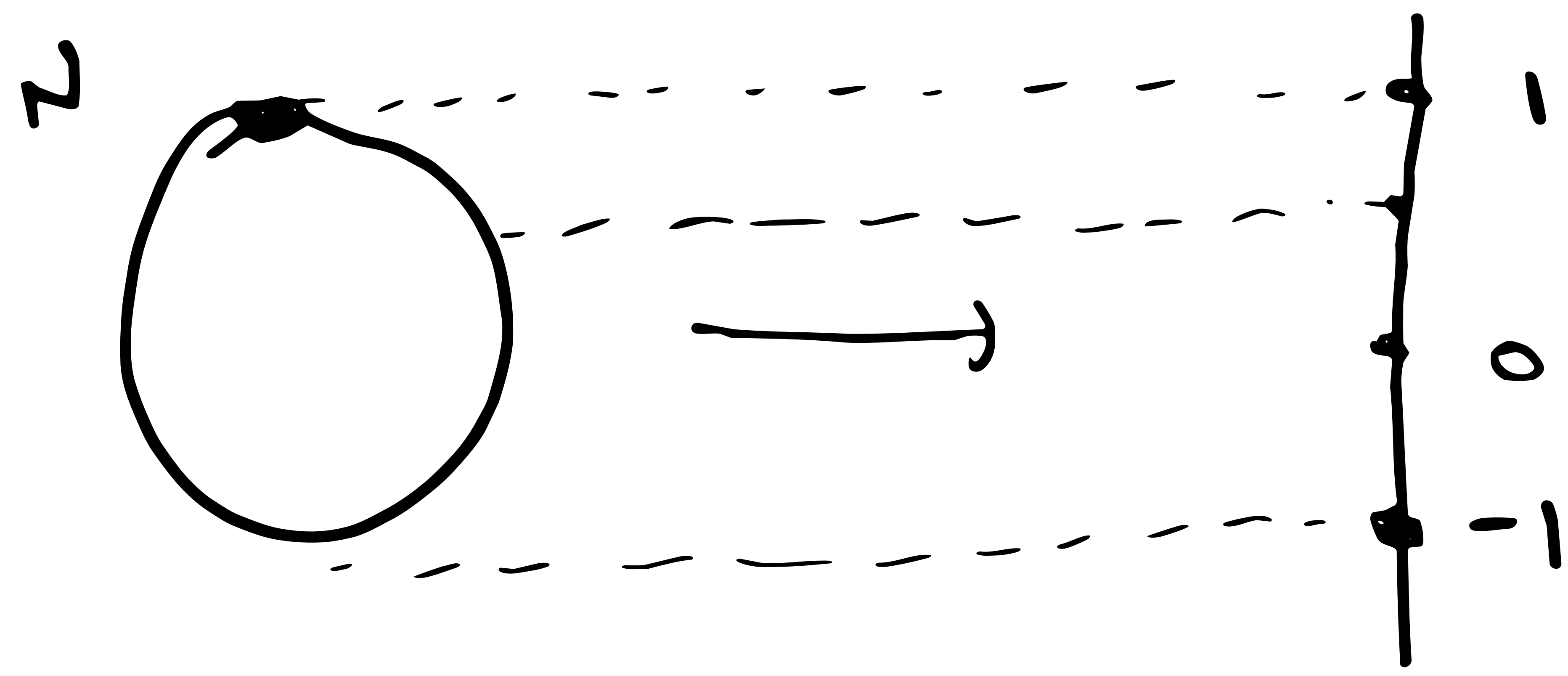
$$f \circ \phi^{-1} : \phi(U) \longrightarrow \mathbb{R}^m$$

is smooth. Then

$$\begin{tikzcd}
\psi(U \cap V) \arrow[r, "\psi^{-1}"] \arrow[rrr, bend right, "f \circ \phi^{-1} \circ \phi \circ \psi^{-1} = f \circ \psi^{-1}"'] & U \cap V \arrow[r, "\phi"] & \phi(U \cap V) \arrow[r, "f \circ \phi^{-1}"] & \mathbb{R}^m \\
\psi(p) \arrow[u, phantom, "\in" rotate=90]
\end{tikzcd}$$

$\blacksquare$

Ex: Let $h : S^1 \to \mathbb{R}$ be the "height" f^n:



We show h is smooth. Using polar coordinates we have chart near $N \in S^1$ (say) defined by

$$\begin{array}{c} U = \{0 < \theta < \pi\} \subset S^1 \\ \phi \big\downarrow \\ \mathbb{R} \end{array} \qquad \phi(\theta) = \theta.$$

Then
$$h \circ \phi^{-1} : (0, \pi) \longrightarrow \mathbb{R}$$
$$t \longmapsto \sin t$$

This is certainly smooth.

Now we generalize to the case of maps

$$f : M^m \longrightarrow N^n$$

where M, N are top^l mflds. We say f is **smooth** if $\forall \; p \in M$ there are charts

M : (U, ϕ) near $p \in M$

N : (V, ψ) near $f(p) \in N$

w/ $f(U) \subset V$

s.t.

$$\begin{tikzcd}
\phi(U) \arrow[r, "\phi^{-1}"] \arrow[d, phantom, "\cap" rotate=90] \arrow[rrrd, "\psi \circ f \circ \phi^{-1}"'] & U \arrow[r, "f"] & V \arrow[r, "\psi"] & \psi(V) \arrow[d, phantom, "\cap" rotate=90] \\
\mathbb{R}^m & & & \mathbb{R}^n
\end{tikzcd}$$

is smooth.

Prop: If $f : M \longrightarrow N$ is smooth then it is continuous.

Pf: know $\psi \circ f \circ \phi^{-1} : \phi(U) \to \mathbb{R}^n$ is smooth hence cts. Now

$$F\big|_U = \psi^{-1} \circ (\psi \circ f \circ \phi^{-1}) \circ \phi : U \to V$$

is composition of cts fn's hence it is cts. Since this is true for some nbd U containing every pt $p \in M$, it follows that f is cts. (this is an easy exercise in topology.) ▨

Smoothness is a "local property" in the following sense.

Prop: $F : M \to N$ is smooth $\Leftrightarrow$ for all $p \in M$ $\exists$ $U \ni p$ open s.t. $F\big|_U : U \to N$ is smooth.

Ex: • Constant fns are smooth.

• If $M \xrightarrow{f} N \xrightarrow{g} P$ are two smooth maps the $g \circ f$ is smooth. (Use chain rule.)

• $f : M \longrightarrow N_1 \times \cdots \times N_k$ is smooth $\Leftrightarrow$

$f \circ \pi_i : M \longrightarrow N_1 \times \cdots \times N_k \xrightarrow{\pi_i} N_i$

is smooth.