September 18

Calculus uses the method of linear approximation to study smooth functions (like the line tangent to the graph of a function $\mathbb{R} \to \mathbb{R}$.)

To make sense of this method for manifolds, we need to define tangent spaces.

In the case of graphs of smooth fn's $\mathbb{R} \to \mathbb{R}$, this is the 1-dimensional vector space which is tangent to the graph at a pt.

We begin w/ open subsets

$$U \subset \mathbb{R}^n.$$

Given $a \in \mathcal{U}$, the tangent space at a is the set

$$T_a\mathcal{U} \overset{\text{def}}{=} \{a\} \times \mathbb{R}^n = \{\underset{\substack{\| \\ v_a}}{(a, v)} \mid v \in \mathbb{R}^n\}.$$

This set is a vector space w/ operations

$$v_a + w_a = (v + w)_a, \quad \lambda v_a = (\lambda v)_a.$$

Of course, as vector spaces

$$T_a\mathcal{U} \cong \mathbb{R}^n,$$

for any $a \in \mathcal{U}$.

We could attempt a similar dfn for <u>closed</u> subsets of $\mathbb{R}^n$, but we won't take that approach.

Let's return to derivatives. Recall

$$\frac{\partial f}{\partial x^j}(a) = \frac{d}{dt} f(a + t e_j)$$

where e_j is the j^{th} unit vector in $\mathbb{R}^n$.

Sps $v \in \mathbb{R}^n$ is any vector. Then we define the v-<u>directional derivative</u> at a:

$$D_v f(a) = \frac{d}{dt} f(a + t v).$$

<u>Lem</u>: 1) $D_v(f+g) = D_v f + D_v g$

2) $D_v(fg) = (D_v f) g + f D_v(g)$

as functions of a.

3) $D_v f(a) = v^i \frac{\partial f}{\partial x^i}(a)$.

where $v = v^i e_i$.

This leads to.

<u>Dfn</u>: Let $a \in \mathcal{U} \overset{\text{open}}{\subset} \mathbb{R}^n$. A

<u>derivation</u> at $a \in \mathcal{U}$ is a map

$$\delta : C^\infty(\mathbb{R}^n) \longrightarrow \mathbb{R}$$

s.t.

1) $\delta(f + \lambda g) = \delta f + \lambda \delta g$

   when $\lambda \in \mathbb{R}$.

2) $\delta(fg) = f(a)\,\delta g + g(a)\,\delta g$.

Let $\mathrm{Der}_a(\mathcal{U})$ be the set of derivations at a. This is naturally a vector space:

$$(\delta_1 + \delta_2) f \overset{\text{dfn}}{=} \delta_1 f + \delta_2 f$$

$$(\lambda \delta) f \overset{\text{dfn}}{=} \lambda \delta f.$$

<u>Lem</u>: Let $a \in U$, $\delta \in \mathrm{Der}_a(U)$ and $f, g \in C^\infty(\mathbb{R}^n)$.

1) If f is constant then $\delta f = 0$.

2) If $f(a) = g(a) = 0$ then $\delta(fg) = 0$.

<u>Prop</u>: The map

$$\begin{aligned} T_a U &\xrightarrow{\cong} \mathrm{Der}_a(U) \\ v_a &\longmapsto \left(f \mapsto D_v f(a)\right) \end{aligned}$$

is a linear isomorphism.

<u>Pf</u>: The map is certainly linear. We show it is injective. Sps $v_a \in T_a U$ is such that

$$\delta_{v_a} : f \longmapsto D_v f(a)$$

is the zero derivation. That is

$$\delta_{v_a} f = 0 \quad \forall \; f \in C^\infty(\mathbb{R}^n).$$

In a basis write $v = v^i e_i$, and consider the coordinate fn:

$$x^i : \mathbb{R}^n \longrightarrow \mathbb{R}.$$

Then by assumption

$$0 = \delta_{v_a}(x^j) = v^i \frac{\partial}{\partial x^i}(x^j) = v^j.$$

$$\Rightarrow \quad v^j = 0 \quad \forall \; j \quad \Rightarrow \quad v = 0$$

Next surjectivity. This follows from the following lemma.

Lem: Let $\frac{\partial}{\partial x^i}$ be the derivation

$$\frac{\partial}{\partial x^i} f = \frac{\partial f}{\partial x^i}(a) .$$

Then $\mathrm{Der}_a(U) = \mathrm{span}\left\{ \frac{\partial}{\partial x^i} \right\}_{i=1}^{n}$.

Pf: By Taylor's theorem

$$f(x) = f(a) + \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(a)\,(x^i - a^i)$$
$$+ \sum_{i,j=1}^{n} (x^i - a^i)(x^j - a^j)\, h_{ij}(x)$$

where $h_{ij}(x)$ is some smooth fn.

For any derivation δ we have

$$\delta\Big((x^i - a^i)(x^j - a^j)\, h_{ij}(x) \Big) = 0$$

by previous lemma. Thus

$$\delta f = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(a) \left(\delta(x^i) - \delta(a^i) \right)$$

$$= \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(a)\, \delta x^i$$

$\Rightarrow$ any δ is a liner combination of $\left\{ \frac{\partial}{\partial x^i} \right\}$. $\blacksquare$

From now on we will freely go between $T_a U$ and $\mathrm{Der}_a(U)$, and will primarily use the first notation.

Now let M be a smooth mfld.
A linear map

$$\delta : C^\infty(M) \longrightarrow \mathbb{R}$$

is a <u>derivation at $p \in M$</u> if

$$\delta(fg) = f(p)\,\delta g + g(p)\,\delta f \,.$$

Denote the vector space of all derivations at p by

$$T_pM = \left\{ \begin{matrix} \text{derivations} \\ \text{at } p \in M \end{matrix} \right\},$$

which we call the <u>tangent space</u> of M at p.

<u>Lem</u>: For $\delta \in T_pM$:

1) $\delta(\text{constant}) = 0$, 2) $f(p) = g(p) = 0 \Rightarrow \delta(fg) = 0$.