September 20

Last time we defined the tangent space at a point in an open subset $U \subset \mathbb{R}^n$.

We now define the tangent space to $p \in M$, w/ M a general manifold of dim $= n$.

<u>Dfn</u>: Let $p \in M$. A <u>derivation</u> at $p \in M$ is a linear map

$$v : C^\infty(\mathbb{R}^n) \to \mathbb{R}$$

s.t. $$v(fg) = vf\, g(a) + f(a) vg.$$

Let $$T_pM = \left\{ \begin{array}{c} \text{set of all} \\ \text{derivations} \\ \text{at } p \in M \end{array} \right\}.$$

call this the tangent space at $p \in M$.

The operations

$$(v_1 + v_2)(f) = v_1 f + v_2 g \qquad f, g \in C^\infty$$

and $$(\lambda v)(f) = \lambda v f \quad , \lambda \in \mathbb{R}$$

make $T_p M$ into a vector space.

Next we move onto the <u>differential</u> of a smooth map.

Let $F : M \longrightarrow N$ be a smooth map between smooth manifolds M, N of dim $= m, n$ resp..

For $p \in M$ define

$$dF_p : T_p M \longrightarrow T_{F(p)} N$$

by the formula

$$(dF_p)(v)(f) = v(f \circ F)$$

Recall from HW $f \circ F$ is denoted $F^* f$, so we can write this like

$$(dF_p)(v)(f) = v(F^* f).$$

[Well-defined: Why is this a derivation?]

<u>Prop</u>: 1) dF_p is a linear map.

2) If $F: M \to N$, $G: N \to P$ then

$$d(G \circ F)_p = dG_{F(p)} \circ dF_p.$$

3) $d\mathbb{1}_p = \mathbb{1}_{T_pM}$ where $\mathbb{1}: M \to M$.

4) $F: M \to N$ diffeomorphism $\Rightarrow$

$$dF_p : T_pM \xrightarrow{\cong} T_{F(p)}N$$

is an isomorphism.

Pf: 2) Let $v \in T_pM$, $f \in C^\infty(P)$:

$$d(G \circ F)_p v(f) = v\left((G \circ F)^* f\right)$$

$$= v\left(F^*(\underbrace{G^* f}_{g})\right)$$

$$= dF_p v(G^* f)$$

$$= dG_{F(p)} \circ dF_p v(f).$$

$$\begin{tikzcd} C^\infty(N) \arrow[r, "F"] \arrow[dr, "dF_p(v)"'] & C^\infty(M) \arrow[d, "v"] \\ & \mathbb{R} \end{tikzcd}$$

3) Use 2).

4) Use 3). $\square$

<u>Lemma</u>: Sps $f, g \in C^\infty(M)$ agree in some nbd of $p \in M$. Then

$$df_p = dg_p$$

as linear maps $T_pM \to T_{F(p)}\mathbb{R} = \mathbb{R}$.

Pf: Let ρ be a smooth "bump" fn on M vanishing at p. That is

$$\rho\Big|_{\mathcal{U}} \equiv 1, \quad \mathcal{U} = \{q \in M \mid f(q) \neq g(q)\}.$$

and $\rho(p) = 0$.

Let $h = f - g$, so $h \in C^\infty(M)$ $h(p) = 0$.

So $\rho h = h$. But then for any $v \in T_pM$ we have

$$v(h) = v(\rho h) = 0 \quad \text{since } \rho(p) = h(p) = 0.$$

$\blacksquare$

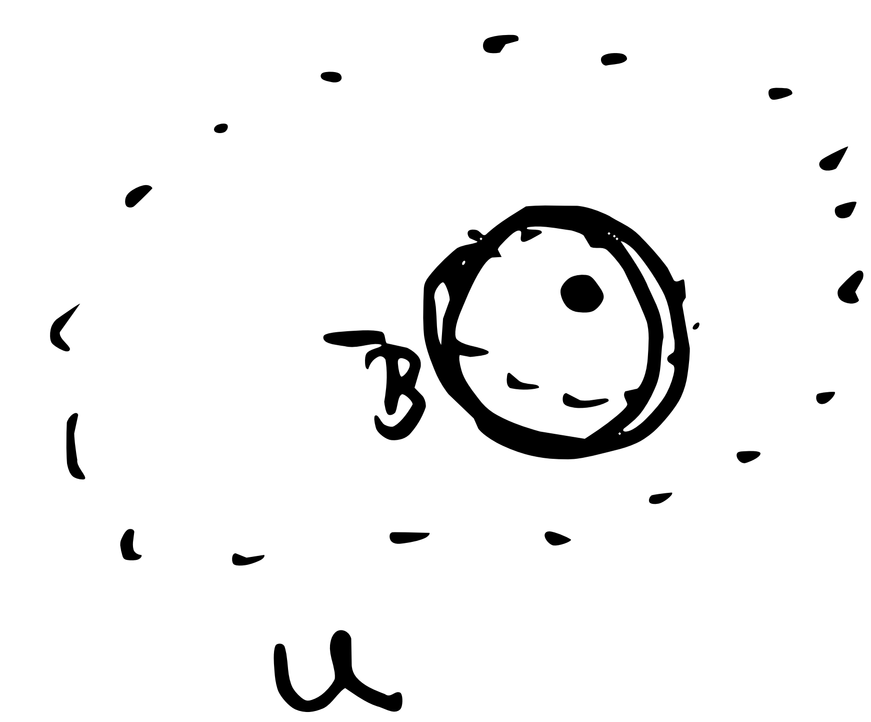
Prop $\mathcal{U} \subset M$ open, let $i : \mathcal{U} \hookrightarrow M$. Then

$$di_p : T_p\mathcal{U} \xrightarrow{\sim} T_pM$$

is an isomorphism.

Pf: Let B be a nbd of $p \in U$ s.t.

$$\bar{B} \subset U.$$



If $f \in C^\infty(U)$ then $\exists\ \tilde{f} \in C^\infty(M)$

s.t. $\tilde{f}|_{\bar{B}} \equiv f_{\bar{B}}$. Since $\tilde{f}|_U$, f

are smooth and agree on B we have

$$v(f) = v\left(\tilde{f}|_U\right) = v\left(i^*\tilde{f}\right) = di_p v(\tilde{f})$$

for any $v \in T_pU$ by Lemma.

Now if $(di_p)v(f) = 0 \quad \forall f \in C^\infty(M)$

$\Rightarrow v(f) = 0 \quad \forall f$

$\Rightarrow v = 0 \Rightarrow di_p$ is injective.

Next we show surjective. Sps $w \in T_p M$.

Define $v_w \in T_p \mathcal{U}$ by

$$v_w f = w \tilde{f} \quad \text{where } \tilde{f} \text{ is some}$$

function on M s.t. $f\big|_{\bar{B}} = \tilde{f}\big|_{\bar{B}}$. By

Lemma this is well-defined, independent

of the choice of extension $\tilde{f}$. Now

for any $g \in C^\infty(M)$ have

$$\begin{aligned} di_p(v_w)(g) &= v_w(g \circ i) \\ &= w(\widetilde{g \circ i}) \\ &= w(g). \end{aligned}$$

$$\Rightarrow \quad di_p(v_w) = w. \qquad \blacksquare$$

Thm: For any $p \in M$:
$$\dim M = \dim T_p M .$$

Pf: Let (U, ϕ) be a smooth chart near p. Then since $\phi : U \xrightarrow{\cong} \phi(U)$ diffeo:

$$d\phi_p : T_p U \xrightarrow{\cong} T_{\phi(p)} \phi(U) \text{ iso.}$$

But we know since $\phi(U) \subset \mathbb{R}^n$ is open $\Rightarrow \dim T_{\phi(p)} \phi(U) = n$.

$\Rightarrow \dim T_p U = n$.

$\Rightarrow \dim T_p M = n$ by last proposition. $\blacksquare$