September 20

Recall that for $\mathcal{U} \subset \mathbb{R}^n$ open that

$$T_p \mathcal{U} \cong \text{span}\left\{ \frac{\partial}{\partial x^i} \right\}.$$

Lemma: Let $F : \mathcal{U} \to \mathbb{R}^m$ be a smooth map. Then for any $a \in \mathcal{U}$.

$$dF_a = DF|_a .$$

total derivative at $a \in \mathcal{U}$.

Pf: Recall in the standard basis that $(DF|_a)_i^j$ is the Jacobian matrix

$$= \frac{\partial F^j}{\partial x^i}.$$

Let $x^j : \mathbb{R}^m \to \mathbb{R}$ be j^{th} coordinate.

$$(dF_a)\left(\frac{\partial}{\partial x^i}\right)(x^j) = \frac{\partial}{\partial x^i}\left(x^j \circ F\right)$$

$$= \frac{\partial F^j}{\partial x^i}. \qquad \boxtimes$$

So, once we identify

$$T_p U \cong \mathbb{R}^n, \quad U \subset_{\text{open}} \mathbb{R}^n$$

we have shown that the differential of a smooth map $F : U \to \mathbb{R}^m$

$$dF_p : T_p U \longrightarrow T_{F(p)} \mathbb{R}^m$$

agrees w/ the total derivative

$$DF\big|_p : \mathbb{R}^n \longrightarrow \mathbb{R}^m.$$

• <u>Coordinates</u> : Sps (U, ϕ) is a chart.

And let $x^i : U \to \mathbb{R}$ be the coordinate fns of ϕ:

$$\phi(p) = (x^1(p), \ldots, x^n(p)) \in \mathbb{R}^n .$$

We know a basis for $T_{\phi(p)} \mathbb{R}^n$ is given by $\left\{ \frac{\partial}{\partial x^i}\Big|_{\phi(p)} \right\}_{i=1}^n$. Since

ϕ is a diffeo onto its image we can identify via

$$d\phi_p : T_pM \xrightarrow{\cong} T_{\phi(p)} \mathbb{R}^n$$

this basis w/ a basis for T_pM that we denote

$$\left\{ \frac{\partial}{\partial x^i}\Big|_p \right\} .$$

Now we can combine **all** of the tangent spaces into a single object.

The **tangent bundle** is the set

$$TM \overset{dfn}{=} \coprod_{p \in M} T_p M .$$

So, as a set TM is the (infinite) disjoint union of vector spaces.

Ex: Since $T_a \mathbb{R}^n \cong \mathbb{R}^n$ we have

$$TM = \mathbb{R}^n \times \mathbb{R}^n .$$

The tangent bundle admits a topology (in fact smooth str.) s.t. the natural map

$$\pi : TM \longrightarrow M$$ is continuous.

Let's describe it.

Sps $(\mathcal{U}, \phi)$ is a local chart for M, and let $\{x^i\}$ be the coordinate fn's:

$$\phi(p) = (x^1(p), \dots, x^n(p))$$

$$x^i : \mathcal{U} \to \mathbb{R}.$$

Define $\tilde{\phi} : \pi^{-1}(\mathcal{U}) \longrightarrow \phi(\mathcal{U}) \times \mathbb{R}^n$ by

$$\tilde{\phi}\left(v^i \frac{\partial}{\partial x^i}\Big|_p\right) = \left(\phi(p) \,;\, v^1, \dots, v^n\right).$$

This is clearly bijective, hence we can transfer the topology from $\phi(\mathcal{U}) \times \mathbb{R}^n$ to one on $\pi^{-1}(\mathcal{U})$.

If we pick a (countable) cover $\{\mathcal{U}_\alpha\}$ of M by coordinate charts, then we get a topology on $\pi^{-1}(U_\alpha)$ for every α. This is Hausdorff and second ctble by construction.

Moreover $\{\pi^{-1}(U_\alpha)\}$ admits coordinate charts

$$\tilde{\phi}_\alpha : \pi^{-1}(U_\alpha) \longrightarrow \phi(U_\alpha) \times \mathbb{R}^n$$

as above. Transition maps:

$$\phi_\beta \circ \phi_\alpha^{-1}(a ; v)$$

$$= \left(\phi_\beta \circ \phi_\alpha^{-1}(a) ; \ J(\phi_\beta \circ \phi_\alpha^{-1})\big|_p v\right).$$

Jacobian.

This is smooth!

We conclude that TM is a smooth manifold.

<u>Prop</u>: Sps M is a smooth manifold equipped w/ a global chart (M, ϕ). Then

$$TM \cong M \times \mathbb{R}^n.$$

<u>Pf</u>: ϕ defines a diffeomorphism of M w/ some open subset $\mathcal{U} \subset \mathbb{R}^n$.
We have just seen that $T\hat{\mathcal{U}} \cong \hat{\mathcal{U}} \times \mathbb{R}^n$. $\boxed{\,}$

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For $F: M \to N$ let

$$dF : TM \longrightarrow TN$$

be defined by $dF(v_p) = dF_p(v)$.

<u>Lem</u>: 1) $dF: TM \longrightarrow TN$ is smooth.

2) $d(G \circ F) = dG \circ dF$

3) $d(\mathbb{1}_M) = \mathbb{1}_{TM}$

4) F diffeo $\Rightarrow$ dF diffeo.

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Velocity vectors and curves.

Let $J \subset \mathbb{R}$ be an open interval.
A (smooth) <u>curve</u> in M is a (smooth) continuous map

$$\gamma : J \longrightarrow M .$$

The <u>velocity</u> of γ at $t_0 \in J$ is the vector

$$\gamma'(t_0) \overset{\text{dfn}}{=} d\gamma_{t_0}\left(\left.\frac{\partial}{\partial t}\right|_{t_0}\right) \in T_{\gamma(t_0)}M .$$

In other words, if $f \in C^\infty(M)$ then

$$\gamma'(t_0) f = d\gamma_{t_0}\left(\frac{d}{dt}\Big|_{t_0}\right) f$$
$$= (f \circ \gamma)'(t_0).$$

ordinary use of this symbol.

<u>Prop</u>: Sps $F : M \to N$ is smooth, $p \in M$, $v \in T_pM$. Then

$$dF_p(v) = (F \circ \gamma)'(0)$$

where $\gamma : J \to M$ is any curve s.t. $0 \in J$, $\gamma(0) = p$ and $\gamma'(0) = v$.