September 29 | Notation as in the last lecture. We have seen $D\phi_0\big|_{(0,0)}$ is invertible.

By inverse fn thm $\exists$ connected nbds

$$\underset{0 \in}{U_0} \subset U, \quad \overset{0\in}{\tilde{U}_0} \subset \mathbb{R}^m \text{ s.t.}$$

$$\phi\big|_{U_0} : U_0 \longrightarrow \tilde{U}_0 \quad \text{is a diffeo.}$$

(By shrinking $U_0, \tilde{U}_0$ we can assume that $\tilde{U}_0$ is a cube: $(0,\varepsilon_1)\times\cdots\times(0,\varepsilon_m)$.)

Write $\phi^{-1}(x,y) = (A(x,y), B(x,y))$

$\Rightarrow B(x,y) = y$. So

$$\phi^{-1}(x,y) = (A(x,y), y)$$

when $A : \tilde{U}_0 \longrightarrow \mathbb{R}^n$.

By dfn $Q(A(x,y),y) = x$.

$\Rightarrow$ $F(\phi^{-1}(x,y)) = (x, \tilde{R}(x,y))$.

where $\tilde{R} : \tilde{U}_0 \longrightarrow \mathbb{R}^{n-r}$ is

$$\tilde{R}(x,y) = R(A(x,y),y).$$

Now

$$D(F \circ \phi^{-1})\Big|_{(x,y)} = \left(\begin{array}{c|c} 1 & 0 \\ \hline \frac{\partial \tilde{R}^i}{\partial x^j}(x,y) & \frac{\partial \tilde{R}^i}{\partial y^j}(x,y) \end{array}\right)$$

is still rank r. on $\tilde{U}_0$.

$\Rightarrow$ $\tilde{R}^i$ is independent of $(y^1, \dots, y^{m-r})$.

Let $S(x) = \tilde{R}(x,0)$.

$\Rightarrow$ $F \circ \phi^{-1}(x,y) = (x, S(x))$.

It remains to find a chart near $F(p) = 0$ s.t. $\hat{F}$ takes the stated form.

Let $V_0 = \left\{ (v,w) \,\middle|\, (v,0) \in \tilde{U}_0 \right\} \subset V \subset \mathbb{R}^n$

Then V_0 is a nbd of 0. Taking $\tilde{U}_0$ small enough we have

$$F \circ \phi^{-1}(U_0) \subset V_0$$

$$\Rightarrow \quad F(U_0) \subset V_0 .$$

Define

$$\psi : V_0 \to \mathbb{R}^n, \quad (v,w) \mapsto (v, w - S(v)) .$$

Then $\psi \circ F \circ \phi^{-1}(x,y) = (x,0)$.

$\blacksquare$

Embeddings

Dfn: A smooth embedding is a smooth map

$$F : M \to N$$

s.t. 1) F is an immersion.

2) $F : M \to F(M)$ is a homeomorphism.

Notation: $F : M \hookrightarrow N$.

Ex: 1) $U \subset M$. The map

$$\mathbb{1}\big|_U : U \to M$$

is an embedding.

2) $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+k}$, $x \longmapsto (x, 0)$ is an embedding.

More generally if $p \in N$ then

$$\begin{array}{ccc} M & \hookrightarrow & M \times N \\ \Psi & & \Psi \\ x & \longmapsto & (x, p) \end{array}$$

is an embedding.

3) $\gamma : \mathbb{R} \longrightarrow \mathbb{R}^2$, $t \longmapsto (t^3, 0)$

is an injective map which is a homeomorphism onto its image. But

$$d\gamma_0 = 0$$

So it is <u>not</u> a smooth embedding.

<u>Prop</u>: $F : M \longrightarrow N$ smooth injective immersion if either

a) M compact or

b) $\dim M = \dim N$

$\Rightarrow$ F is smooth embedding.

Before proving this we collect some topology.

We say F is open/closed if

$$S \subset M \text{ open/closed} \Rightarrow F(S) \subset N \text{ open/closed.}$$

Lem: If $F: M \to N$ is open or closed and injective then F is a homeomorphism onto its image.

Pf: Assume F is open.

By assumption $F: M \to F(M)$ is bijective. If $U \subset M$ is open then

$$(F^{-1})^{-1}(U) = F(U) \subset N$$

is open by assumption. This shows that F^{-1} is cts.

Now suppose F is closed. A subset

$$S \subset F(M)$$

is closed iff $\exists$ closed $K \subset N$

s.t. $S = K \cap F(N)$. Now if $T \subset M$ is closed

$$(F^{-1})^{-1}(T) = F(T) \subset F(N) \subset N$$

is closed by assumption. $\boxed{/ / /}$

We now return to proposition.

Pf: By the lemma, if F is open or closed (in addition to being an injective smooth immersion) then it is a smooth embedding.

If M is compact then any cts fn $M \to N$ w/ N Hausdorff is automatically closed. So this gives (a).

(b) From last time we know that if dF_p is invertible every point then it is a local diffeomorphism.
$\Rightarrow$ F is an open map. $\boxtimes$

<u>Ex</u>: We have a natural map:

$$i : S^n \hookrightarrow \mathbb{R}^{n+1}.$$

This is a smooth embedding as one can show di_p is injective for all $p \in S^n$, and S^n is compact.

The following result implies that immersions are "locally" embeddings.

<u>Thm</u>: $F : M \to N$ is an immersion
$\iff$ $\forall\, p \in M$ $\exists\, U \ni p$ nbd s.t.

$$F|_U : U \hookrightarrow N \text{ is smooth embedding.}$$