September 6

Welcome to MA 721. The course starts w/ a brief excursion into the world of topological manifolds.

A top^l manifold is, roughly, a top^l space (henceforth just called "space") which looks like $\mathbb{R}^n$, locally.

This is not a place where we can speak of derivatives, like in calculus, but its a step along the way.

Next week, we will introduce the concept of a smooth structure which will allow us to generalize all of those familiar ideas in calculus/analysis.

First, here is a rapid review of topology.

- A top^{l} space is a set X equipped w/ a collection of <u>open</u> subsets s.t.
  1) $\emptyset, X$ are open.
  2) $\bigcup_{\alpha} U_\alpha$ open if $\{U_\alpha\} \subset X$ open.
  3) $\bigcap_{i=1}^{N} U_i$ open if $\{U_i\}_{i=1}^{N} \subset X$ open.

- A function $f: X \to Y$ is <u>continuous</u> if $f^{-1}(U)$ is open for all open $U \subset Y$.

- A <u>homeomorphism</u> is a cts isomorphism f s.t. f^{-1} is cts.

- The <u>subspace</u> topology on a subset $Y \subset X$ of a space is s.t. $U \subset Y$ open $\iff$ $\exists\ \tilde{U} \subset X$ open s.t. $U = \tilde{U} \cap Y$.

Here are some further ideas from topology we will use this week. (By "space" we will mean a topological space.)

We say a space M is:

1) Hausdorff: if $\forall$ $x \neq y \in M$ there are opens $\mathcal{U}, V \subset M$ s.t.

$$x \in \mathcal{U}, \quad y \in V, \quad \mathcal{U} \cap V = \emptyset.$$

2) Second countable: if $\exists$ a countable topological basis for M.

3) Locally Euclidean: if $\forall$ $x \in M$ $\exists$ a nbd $\mathcal{U}$ of x s.t.

$$\mathcal{U} \underset{\text{homeomorphic}}{\cong} \hat{\mathcal{U}}.$$

where $\hat{U}$ is an open subset $\hat{U} \subset \mathbb{R}^n$ for some n. (We require that n be the same for all $x \in M$).

Dfn: A topological manifold is a space satisfying ① - ③ above.

Perhaps the most important property is ③; let's unpack it. We say a coordinate chart at $x \in M$ is a pair

$$(U, \varphi : U \longrightarrow \mathbb{R}^n)$$

where

- $U \ni x$ is an open subset, containing the point x.
- $\varphi : U \longrightarrow \mathbb{R}^n$ is a cts map s.t. $\varphi : U \xrightarrow{\cong} \varphi(U)$ is a homeomorphism.

Thus (3) is equivalent to the existence of coordinate charts at every $x \in M$.

A coordinate chart gives local coordinates $\{x^i\}_{i=1}^n$ where $x^i : U \to \mathbb{R}$ are :

$$\varphi(p) = \left(x^1(p), \ldots, x^n(p) \right) \in \mathbb{R}^n .$$

<u>Ex</u>: $\mathbb{R}^n$, and any open subset of $\mathbb{R}^n$ is a top^l manifold.

<u>Ex</u>: $S^n = \left\{ x \in \mathbb{R}^{n+1} \,\middle|\, \|x\| = 1 \right\}$ is a

top^l manifold. Being a subspace of $\mathbb{R}^{n+1}$ it is automatically Hausdorff and 2^{nd} ctble. We now construct coordinate charts. Let

$$U^{\pm} = \left\{ (x^1 \ldots x^{n+1}) \,\middle|\, x^{n+1} \gtrless 0 \right\}.$$

and define the charts

$$\varphi^{\pm} : U^{\pm} \cap S^n \longrightarrow \mathbb{R}^n$$

$$(x^1, \dots, x^{n+1}) \longmapsto (x^1, \dots, x^n) .$$

* check that this is a homeomorphism onto its image.

More examples.

Prop: If M, N are top^l manifolds, then so is $M \times N$.

Pf: Sps $(p, q) \in M \times N$, and let

$$\varphi : \underset{\substack{\cup\!\!\!\!\!\!\shortmid \\ p}}{U} \longrightarrow \mathbb{R}^m$$

$$\psi : \underset{\substack{\cup\!\!\!\!\!\!\shortmid \\ q}}{V} \longrightarrow \mathbb{R}^n$$

be coordinate charts for M, N.

Then

$$\varphi \times \psi : \mathcal{U} \times V \longrightarrow \mathbb{R}^{n+m}$$

is a coordinate chart for $(p,q) \in M \times N$.

$\boxed{\nabla}$