September 8

Today we begin to introduce smooth structures on a topological manifold. This will allow us to port all ideas from calculus to the context of manifolds.

In calculus, we typically study functions

$$f : U \longrightarrow \mathbb{R}, \quad U \subset \mathbb{R}^n.$$

We say f is *smooth* if all partial derivatives

$$\partial_{x_i}^k f : U \longrightarrow \mathbb{R} \quad \forall\, i, k.$$

exist and are continuous. (We will review this concept shortly)

Given a top^l manifold M what does it mean for a fn

$$f : M \longrightarrow \mathbb{R}$$

to be smooth? (Think about what sort of properties smoothness should have)

<u>Attempt</u>: We know M admits a cover by coordinate charts. If $p \in M$, let $(U \ni p, \phi)$ be a chart

$$\phi : U \longrightarrow \mathbb{R}^n$$

Thus the composition

$$\phi(U) \xrightarrow{\ \phi^{-1}\ } U \xrightarrow{\ f\ } \mathbb{R}$$

$$\phi(U) \xrightarrow{\ f \circ \phi^{-1}\ } \mathbb{R}, \qquad \phi(U) \subset \mathbb{R}^n$$

is of the form we studied in calculus. So, one defn of "smooth" would be to say that $f \circ \phi^{-1}$ is smooth for any coordinate chart (U, ϕ).

Problem: Why is this independent of the chosen chart?

In general it is not! So we need some refinement of a covering by charts ...

Let's give a precise definition. Sps M is a top^l manifold and let

$$(U, \phi), (V, \psi) \qquad U \cap V \neq \phi$$

be charts. The transition map is the composition:

$$\phi(U\cap V) \xrightarrow{\phi^{-1}} U\cap V \xrightarrow{\psi} \psi(U\cap V)$$

$$\mathbb{R}^n \supset \phi(U\cap V) \xrightarrow{\psi\circ\phi^{-1}} \psi(U\cap V) \subset \mathbb{R}^n.$$

We say (U,ϕ), (V,ψ) are <u>smoothly compatible</u> if $\psi\circ\phi^{-1}$ is a smooth homeomorphism

$\|$

diffeomorphism.

First we review some ideas from calculus, which will equip us with all of the 'local' tools that we will need.

This will allow us to formulate what "smooth" means locally and next time we will globalize this dfn to manifolds.

We will use some elementary linear algebra and a bit of real analysis. For more of a review I recommend appendix B from Lee's book.

* <u>Warning</u>: We will use some arguments familiar to you if you've taken real analysis. If you need to review this, consult Appendix of Lee.

Let V, W be finite-dimensional normed vector spaces. For an open set $\mathcal{U} \subset V$ a map

$$F : \mathcal{U} \to W$$

is said to be differentiable at $a \in \mathcal{U}$

if $\exists$ a linear map $L : V \to W$ s.t.

$$\lim_{v \to 0} \frac{|F(a+v) - F(a) - Lv|}{|v|} = 0 .$$

If F is differentiable, then the linear map L is actually unique, we denote

$$L = DF(a) ,$$

and we call this the total derivative of F at $a \in \mathcal{U}$.

Many standard properties:

- F is differentiable at $a \Rightarrow F$ is continuous at a.
- If $F \equiv c \in \mathbb{R}$, constant function, then $DF(a) = 0$ for all $a \in \mathcal{U}$.
- If $F, G : \mathcal{U} \to W$ are d'ble at a, then so is $F + G$, and

$$D(F+G)(a) = DF(a) + DG(a).$$

- If $W = \mathbb{R}$, then this notion agrees w/ differentiability for functions. In particular, chain and product rule hold in this case.

There is the following generalization of the chain rule.

<u>Prop</u>: [Chain rule] Suppose

$$\begin{array}{ccccc} & V & & W & & X \quad \text{vector spaces.} \\ \text{open} & \cup & \text{open} & \cup & \nearrow & \\ & \mathcal{U} & \xrightarrow[F]{} & \tilde{\mathcal{U}} & G & \end{array}$$

where
- F differentiable at $a \in \mathcal{U}$
- G differentiable at $F(a) \in \tilde{\mathcal{U}}$.

Then $G \circ F$ is differentiable at $a \in \mathcal{U}$, and

$$D(G \circ F)(a) = DG(F(a)) \circ DF(a).$$

↑ composition of linear maps.

$V \longrightarrow W \longrightarrow X$.

Pf: Let $A = DF(a)$, and $B = DG(b)$.
$b = F(a)$.

It suffices to prove that

$$\frac{|G(F(a+v)) - GF(a) - BAv|}{|v|} \xrightarrow{v \to 0} 0.$$

Re-arrange:

$$F(a+v) = b + (\overbrace{F(a+v) - b}^{w}),$$

so the numerator above is

$$|G(b+w) - G(b) - BAv|$$

$$\leq |G(b+w) - G(b) - Bw| + |B(w - Av)|.$$

Δ-inequality

Since linear maps between finite dim^{l} v.s.'s are always uniformly bounded, we have

$$|Ax| \leq c_A |x| \ , \ |By| \leq c_B |y|$$

for some $c_A, c_B > 0$.

Also since F is d'ble at $a \in \mathcal{U}$ we know that for all $\varepsilon > 0$ there is a nbd of $0 \in V$ s.t.

$$|w - Av| \leq \varepsilon |v| \ , \text{ for } v \text{ in this nbd.}$$

Next, since F is cts at $a \in \mathcal{U}$, have

$$|w| \xrightarrow{v \to 0} 0 \ ,$$

so by taking a possibly smaller nbd of 0, we can ensure that

$$|G(b+w) - G(b) - Bw| \leq \varepsilon |w|$$

by d'blity of G.

Combining this, we see

$$\left(|G(b+w) - G(b) - Bw| + |B(w - Av)| \right) \cdot \frac{1}{|v|}$$

$$\lesssim \varepsilon \frac{|w|}{|v|} + c_B \frac{|w - Av|}{|v|}$$

$$= \varepsilon \frac{|w - Av + Av|}{|v|} + c_B \frac{|w - Av|}{|v|}$$

$$\lesssim (\varepsilon + c_B) \frac{|w - Av|}{|v|} + \varepsilon \frac{|Av|}{|v|}$$

$$\lesssim (\varepsilon + c_B)\, \varepsilon + c_A \varepsilon \longrightarrow 0 \ . \quad \blacksquare$$

Now we move on to <u>partial</u> derivatives and the notion of continuous differentiability.

Sps $\mathcal{U} \subset \mathbb{R}^n$ open, and

$$f : \mathcal{U} \longrightarrow \mathbb{R} \quad \text{a function.}$$

The j^{th} <u>partial derivative</u> of f at $a \in \mathcal{U}$ (if it exists) is the number

$$\frac{\partial f}{\partial x_j}(a) \overset{\text{def}}{=} \lim_{h \to 0} \frac{f(a + he_j) - f(a)}{h}$$

where $e_j = (0 - \overset{j}{\overbrace{1}} - 0)$. j^{th} unit vector.

More generally, if $F : \mathcal{U} \longrightarrow \mathbb{R}^m$, then

$$F(x) = (F^1(x), \dots, F^m(x)).$$

Have partial derivatives of each component function F^i, $i = 1, \dots, m$:

$$\frac{\partial F^i}{\partial x^j}(a), \qquad i = 1, \dots, m \quad j = 1, \dots, n .$$

We call the resulting $m \times n$ matrix $JF(a)$

$$(JF)^i_j(a) = \frac{\partial F^i}{\partial x^j}(a)$$

the <u>Jacobian matrix</u> at $a \in U$

If

$$F : U \longrightarrow \mathbb{R}^m$$

show this is the same notion as before.

is such that all partials exist at $a \in U$, we say F is d'ble at a. If d'ble at <u>all</u> $a \in U$

and $\frac{\partial F^i}{\partial x^j} : U \longrightarrow \mathbb{R}$ are cts,

then we say F is continuously differentiable.

$$C^1(U, \mathbb{R}^m) = \left\{ \begin{array}{c} \text{space of all ctsly d'ble} \\ F : U \longrightarrow \mathbb{R}^m . \end{array} \right\}$$

Can iterate this to get $C^k(U, \mathbb{R}^m)$,
for any integer $k > 0$.

Defn: • We say F is **smooth** if it is of class C^k for all $k \geq 0$.

• A **diffeomorphism** is a map

$$F : U \longrightarrow V, \quad U \subset \mathbb{R}^n, \quad V \subset \mathbb{R}^m$$

which is [1] smooth, [2] bijective, and [3] F^{-1} is smooth.

Prop: Sps $F : U \to \mathbb{R}^m$ is d'ble at $a \in U$.
Then

$$DF(a) = JF(a)$$

as linear maps $\mathbb{R}^n \longrightarrow \mathbb{R}^m$.

Pf: Let $J = DF(a)$, and

$$R(v) = F(a+v) - F(a) - Jv$$

when v is "small" so that $a+v \in U$.

$$F \text{ d'ble} \implies \frac{R(v)}{|v|} \xrightarrow{v \to 0} 0.$$

Also

$$\frac{\partial F^i}{\partial x^j}(a) = \lim_{t \to 0} \frac{F^i(a + t e_j) - F^i(a)}{t}$$

$$= \lim_{t \to 0} \frac{J^i_j t + R^i(t e_j)}{t}$$

$$= J^i_j + \lim_{t \to 0} \frac{R^i(t e_j)}{t}$$

$$= J^i_j.$$

∎

A useful variation of partial derivatives is the directional derivative.

Sps $U \subset \mathbb{R}^n$ and $F : U \to \mathbb{R}^m$.

For fixed $v \in \mathbb{R}^n$, the

<u>v-directional derivative</u> of F at $a \in U$

is

$$D_v F(a) = \left.\frac{d}{dt}\right|_{t=0} F(a + tv).$$

<u>Rmk</u>: $$\frac{\partial F^i}{\partial x^j}(a) = D_{e_j} F^i(a).$$

<u>Prop</u>: Sps $F : U \to V \subset \mathbb{R}^m$ is smooth. Then F is diffeomorphism $\Rightarrow DF(a) = JF(a)$ is a linear isomorphism for all $a \in U$.

Pf: Let F^{-1} be inverse. Note that

if $\mathbb{1}_U : U \to U$ then

$$(D\mathbb{1}_U)(a) = \mathbb{1}_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n .$$

By chain rule

$$\mathbb{1} = D(F^{-1} \circ F)(a)$$

$$= (DF^{-1})(F(a)) \circ DF(a)$$

$\Rightarrow$ $DF(a)$ invertible w/

$$DF(a)^{-1} = (DF^{-1})(F(a)) .$$

$\blacksquare$