HOMEWORK 1

DUE ON FEBRUARY 14

(1) Let $E \to M$ be a vector bundle on a Riemannian manifold M. Show that a differential operator $H\colon \mathcal{E} \to \mathcal{E}$ is a generalized Laplacian if and only if

$$[[H, f], f] = -2|\mathrm{d}f|^2$$

for all $f \in C^\infty(M)$.

(2) Let (M, g) be an n-dimensional Reimannian manifold and let $x_0 \in M$. In this problem, you will show that in the normal coordinate system one has

$$g(\mathbf{x}) = \mathbb{1} + O(|\mathbf{x}|^2),$$

by the following steps.

There are two local frames for the tangent bundle we will use: i) the first comes from the smooth structure on M and will be denoted $\{\partial_i\}$, ii) the second is the orthonormal frame $\{e_i\}$ obtained by radial parallel transport of the orthonormal frame $\{\partial_i|_{x_0}\}$ of $T_{x_0}M$ (in particular, $\{e_i|_x\}$ is an orthonormal frame of $\mathrm{T}_x M$ for all $x \in M$ where the coordinate is defined).

(a) In the frame $\{\partial_i\}$ the *fundamental one-form* is, by definition, the local section $\sum_i \mathrm{d}x^i \partial_i$ of $\mathrm{T}^*_M \otimes \mathrm{T}_M$. Show that this is the local expression of a unique globally defined section $\theta \in \Omega^1(M, \mathrm{T}_M)$ which satisfies $i_X\theta = X$ for all vector fields X.

(b) Write θ in the frame $\{e_i\}$ as $\theta = \sum_j \theta^j e_j = \sum_{i,j} \theta^j_i \mathrm{d}x^i e_j$ and let ω be the $\mathfrak{gl}(n)$-valued one-form of the Levi–Civita connection in the frame $\{e_i\}$. Show that

$$\mathrm{d}\theta + \omega \wedge \theta = 0,$$

and $g_{ij}(\mathbf{x}) = \sum_k \theta^k_i \theta^k_j$ in the frame $\{e_i\}$.

(c) Let $Eu = \sum_i x^i \partial_i$ be the Euler vector field defined in the frame $\{\partial_i\}$. Show that in the frame $\{e_i\}$ that one has $Eu = \sum_i x^i e_i$. Conclude that in the frame $\{e_i\}$ one has

$$i_{Eu}\theta = \mathbf{x} = (x^1, \ldots, x^n).$$

(d) In class we have seen that in the coordinate system $\{e_i\}$ that $L_{Eu}\omega = \iota_{Eu}\Omega$ where Ω is the curvature of the Levi–Civita connection ω. Show that as vector valued one-forms one has

$$(L_{Eu} - 1)L_{Eu}\theta = (\iota_{Eu}\Omega)\mathbf{x}.$$

(e) Conclude that in the frame $\{e_i\}$ one has

$$g_{ij}(\mathbf{x}) = \delta_{ij} + O(|\mathbf{x}|^2),$$

and that the higher order terms can be expressed in terms of the Taylor coefficients of the Riemann curvature $R_{ijkl} = (\Omega(\partial_k, \partial_l)\partial_j, \partial_i)|_{\mathbf{x}}$.