HOMEWORK 2

DUE ON MARCH 6

There are **three** problems to turn in.

(1) Let M be a compact Riemannian manifold and suppose E_1, E_2, E_3 are vector bundles on M. Suppose that

$$p_1 \in \Gamma\left(M \times M, (E_2 \otimes \mathrm{Dens}^{1/2}) \boxtimes (E_1^* \otimes \mathrm{Dens}^{1/2})\right)$$
$$p_2 \in \Gamma\left(M \times M, (E_3 \otimes \mathrm{Dens}^{1/2}) \boxtimes (E_2^* \otimes \mathrm{Dens}^{1/2})\right)$$

are kernels. Let $P_1 \colon \Gamma(M, E_1 \otimes \mathrm{Dens}^{1/2}) \to \Gamma(M, E_2 \otimes \mathrm{Dens}^{1/2})$ and $P_2 \colon \Gamma(M, E_2 \otimes \mathrm{Dens}^{1/2}) \to \Gamma(M, E_3 \otimes \mathrm{Dens}^{1/2})$ be the corresponding operators. Show that the operator $P_2 \circ P_1$ is associated to the kernel

$$\int_{z \in M} p_2(x,z) p_1(z,y) \in \Gamma\left(M \times M, (E_3 \otimes \mathrm{Dens}^{1/2}) \boxtimes (E_1^* \otimes \mathrm{Dens}^{1/2})\right).$$

(2) Consider the heat kernel on $\mathbf{R}^n$

$$q_t(x,y) \stackrel{\mathrm{def}}{=} \frac{1}{(4\pi t)^{n/2}} e^{-\|x-y\|^2/4t},$$

which is defined for all $t > 0$ and $x, y \in \mathbf{R}^n$. Show that

$$\int_{z \in \mathbf{R}^n} q_t(x,z) q_s(z,y)\, \mathrm{d}^n z$$

exists for all $t, s > 0$ and equals $q_{t+s}(x,y)$.

(3) Let $q_t(x,y)$ be as in the previous problem. Consider the n-form on $\mathbf{R}^n \times \mathbf{R}^n$:

$$k_t(x,y) \stackrel{\mathrm{def}}{=} q_t(x,y)(\mathrm{d}^n x - \mathrm{d}^n y) \in \Omega^n(\mathbf{R}^n \times \mathbf{R}^n).$$

(a) Let d^* be the adjoint to the de Rham operator on $\mathbf{R}^n$. Show that

$$\omega \stackrel{\mathrm{def}}{=} \int_{t=0}^{\infty} (\mathrm{d}^* \otimes \mathbb{1})\, k_t(x,y) \mathrm{d}t$$

is a smooth $(n-1)$-form on $\mathbf{R}^n \times \mathbf{R}^n$ away from the diagonal.

(b) Show that the smooth $(n-1)$ form $\omega \in \Omega^{n-1}(\mathbf{R}^n \times \mathbf{R}^n \setminus \mathrm{diag})$ is the pullback of the volume form on S^{n-1} along the projection $\pi \colon \mathbf{R}^n \times \mathbf{R}^n \setminus \mathrm{diag} \to S^{n-1}$ defined by

$$\pi(x,y) = \frac{x-y}{\|x-y\|}.$$

(c) Consider the distributional form

$$\omega_L(x,y) \stackrel{\text{def}}{=} \int_{t=0}^{L} (\mathrm{d}^* \otimes \mathbb{1})\, k_t(x,y)\mathrm{d}t.$$

Show that

$$\mathrm{d}\omega_L(x,y) = \delta(x-y) + \text{smooth}$$

where "smooth" denotes some smooth n-form.