HOMEWORK 3

DUE ON APRIL 18

(1) Consider the Clifford algebra $C(\mathbf{R}^3)$ associated to n-dimensional Euclidean space.
   (a) Construct an isomorphism
$$C(\mathbf{R}^3) \cong M_2(\mathbf{C})$$
   as *real* algebras.
   (b) Recall that group $Spin(3)$ is, by definition, the exponential of the quadratic elements in $C(\mathbf{R}^3)$. Using (a), show that $Spin(3)$ is isomorphic to the group of unit quaternions.
   (c) Show that the group homomorphism $Spin(3) \times Spin(3) \to SO(4)$ which sends (g, h) to the orthogonal transformation $q \mapsto gq\overline{h}$ is a double cover. Conclude that $Spin(4) \cong Spin(3) \times Spin(3)$.

(2) This problem concerns the signature operator defined on compact Riemannian manifold.
   (a) Let V be an oriented Euclidean vector space and $C(V)$ the resulting Clifford algebra and $C(V)_{\mathbf{C}} = C(V) \otimes_{\mathbf{R}} \mathbf{C}$ its complexification. If $\{e_i\}$ is an oriented, orthonormal basis for V define the operator
$$\Gamma = \mathrm{i}^p e_1 \cdots e_n \in C(V)_{\mathbf{C}}, \tag{1}$$
   where $p = n/2$ if n is even and $(n+1)/2$ if n is odd. Show that $\Gamma^2 = 1$.
   (b) Let M be a compact, oriented, Riemannian manifold of dimension n and let
$$\star \colon \Omega^k(M) \to \Omega^{n-k}(M) \tag{2}$$
   be the action by Γ via the standard Clifford module structure on $\wedge^\bullet \mathrm{T}^*_M \otimes \mathbf{C}$ (here $\Omega^i(M)$ denotes sections of the complexified bundle $\wedge^i \mathrm{T}^*_M \otimes \mathbf{C}$). (Note that this definition of $\star$ is NOT the usual Hodge $*$-operator, since $\star^2 = 1$ by construction.) Let d^* be the adjoint to the de Rham operator defined by the metric. Show that
$$\mathrm{d}^* = (-1)^{n+1} \star \mathrm{d} \star. \tag{3}$$
   We call $\mathrm{d} + \mathrm{d}^*$ the *signature operator*.

(c) Assume that n is even. Define a $\mathbf{Z}/2$ grading on $\Omega^\bullet(M)$ as

(4) $$\Omega^\bullet(M)^\pm \overset{\text{def}}{=} \{\alpha \in \Omega^\bullet(M) \mid \star\alpha = \pm\alpha\}.$$

This endows $\wedge^\bullet \mathrm{T}_M \otimes \mathbf{C}$ with a non-standard complex Clifford module structure (since we are changing what the underlying super structure is). Show that when n is divisible by four, that this defines a non-standard Clifford module structure on the (real) bundle $\wedge^\bullet \mathrm{T}_M$ and that the signature operator is a Dirac operator for this Clifford module.

- Assume that n is divisible by four for the remainder of the problem. Suppose we have a quadratic form Q on a real Euclidean vector space V equipped with a basis such that

(5) $$Q(x) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2.$$

The *signature* of Q is defined to be $\sigma(Q) = p - q$. Define the bilinear form on $H^{n/2}(M, \mathbf{R})$ by $\langle \alpha, \beta \rangle = \int_M \alpha \wedge \beta$ (notice this is symmetric because n is divisible by four). Let $\sigma(M)$ be the signature of the resulting quadratic form.

(d) Show that if $k < n/2$ that

(6) $$\operatorname{ind}\left((\mathrm{d} + \mathrm{d}^*)|_{\Omega^k \oplus \Omega^{n-k}}\right) = 0.$$

Conclude that

(7) $$\operatorname{ind}(\mathrm{d} + \mathrm{d}^*) = \sigma(M).$$

Thus, by the McKean–Singer theorem we have that

(8) $$\sigma(M) = \operatorname{Tr}(e^{-t\Delta}),\ t > 0$$

where Δ is the standard Laplace–Beltrami operator.