THE MAIN THEOREM

Dirac's goal was to find a first-order differential operator whose square is the Laplacian. A ***generalized Laplacian*** H is a second order differential operator acting on sections of a vector bundle E over a Riemannian manifold M with the property that its symbol evaluated at $(x, \xi) \in M \times T_x^*M$ is $|\xi|^2$. In the same spirit as Dirac; Berline, Getzler, and Vergne define a Dirac operator to be any differential operator whose square is a generalized Laplacian.

Definition 0.1. Let $E = E^+ \oplus \Pi E^-$ be a super vector bundle on a Riemannian manifold M. A ***Dirac operator*** on E is an odd first-order differential operator

$$\mathsf{D} \colon \mathcal{E} \to \mathcal{E} \tag{1}$$

such that D^2 is a generalized Laplacian.

A fundamental result is that if M is compact then a Dirac operator D on M has finite dimensional kernel. The Atiyah–Singer index theorem is an expression for the index

$$\operatorname{ind} \mathsf{D} = \dim\ker \mathsf{D}^+ - \dim\ker \mathsf{D}^-. \tag{2}$$

In other words, the index is the super-dimension of $\ker D$. To state the index theorem it is convenient to assume that we have a Dirac operator associated to a so-called Clifford module structure on the bundle E. (We will see that this is at no loss of generality, there is a one-to-one correspondence between Clifford module structures and compatible Dirac operators.)

These notes sketch the proof of the following index theorem as presented in the book of Berline, Getzler, and Vergne.

Theorem 0.2. *Let* D *be the Dirac operator associated to a Clifford module* $\mathcal{E}$ *over a compact oriented manifold* M *of even dimension. Then*

$$\operatorname{ind}(\mathsf{D}) = \frac{1}{(2\pi \mathrm{i})^{n/2}} \int_M \widehat{A}(M)\,\mathrm{ch}(\mathcal{E}/S). \tag{3}$$

1. Heat kernels of generalized Laplacians and their trace

Let E be a vector bundle on a Riemannian manifold M. Let Dens^s be the bundle of s-densities on M; this is the line bundle associated to the one-dimensional representation $|\det|^{-s}$. A ***kernel*** is a section

$$k(x,y) \in \Gamma\big(M \times M, (E^* \otimes \mathrm{Dens}^{1/2}) \boxtimes (F \otimes \mathrm{Dens}^{1/2})\big). \tag{4}$$

A kernel determines an operator

$$K\colon \overline{\Gamma}_c(M, E \otimes \mathrm{Dens}^{1/2}) \to \Gamma(M, F \otimes \mathrm{Dens}^{1/2}) \tag{5}$$

defined by the formula $(Ks)(x) = \int_{y \in M} k(x,y)s(y)$. Here $\overline{\Gamma}(M, -)$ denotes distributional (or generalized) sections. The Schwarz kernel theorem asserts an equivalence between bounded linear operators of the above type and kernels. If K is an operator of this type, we will often write the associated kernel as $\langle x|K|y\rangle$.

We are most interested in making sense of the $\mathbf{R}_+$-family of operators e^{-tH} where H is a generalized Laplacian. A ***heat kernel*** $p_t(x,y)$ axiomatizes the properties that the kernel of such a family of operators must possess. A heat kernel $p_t(x,y)$ for H is of class C^1 in t, of class C^2 in x, y. Importantly, a heat kernel satisfies the heat equation

$$(\partial_t + H_x)p_t(x,y) = 0 \tag{6}$$

together with the initial condition $\lim_{t\to 0} p_t(x,y) = \delta(x-y)$.

On Euclidean space $\mathbf{R}^n$, there is the following explicit expression for the heat

$$q_t(x,y) = \frac{1}{(4\pi t)^{n/2}} e^{-\|x-y\|^2/4t}. \tag{7}$$

To produce the heat kernel associated to an arbitrary generalized Laplacian H one proceeds by the following steps.

(1) First, one constructs a *formal* heat kernel of the form

$$k_t(x,y) = q_t(x,y) \sum_{i=0}^{\infty} t^i \Phi_i(x,y,H) |\mathrm{d}y|^{1/2} \tag{8}$$

By formal one means a few things. The sections Φ_i are defined only in a neighborhood of the diagonal in $M \times M$, and the resulting local section $x \mapsto \Phi_t(x,y)$ satisfies the modified heat equation

$$(\partial_t + t^{-1}\nabla_{Eu} + j^{1/2} \circ H \circ j^{-1/2})\Phi_t(\cdot, y) = 0, \tag{9}$$

where Eu is the Euler vector field defined using normal coordinates in a neighborhood of y, and j is the determinant of the Jacobian matrix in normal coordinates.

(2) From a formal solution $k_t(x,y)$ one uses a cut-off function $\psi\colon \mathbf{R}_+ \to [0,1]$ to define an *approximate* solution of the form

$$k_t^N(x,y) = \psi(d(x,y)^2) q_t(x,y) \sum_{i=0}^{N} t^i \Phi_i(x,y,H) |\mathrm{d}y|^{1/2} \tag{10}$$

which is defined everywhere on $M \times M$ and for each $N \geq 0$. The key property of the approximate heat kernel is that its failure to satisfy the heat equation

$$r_t^N(x,y) \stackrel{\text{def}}{=} (\partial_t + H_x)k_t^N(x,y) \tag{11}$$

satisfies an estimate of the form

$$\|r_t^N(x,y)\|_\ell \le C(\ell)t^{N-n/2-\ell/2} \tag{12}$$

for each $\ell > 0$.

(3) From the approximate solution one defines a family of kernels

$$q_t^{N,k}(x,y) \stackrel{\text{def}}{=} \int\limits_{t\Delta^k}\int\limits_{M^k} k_{t-t_k}^N(x,z_k)r_{t_k-t_{k-1}}(z_k,z_{k-1})\cdots r_{t_1}(z_1,y) \tag{13}$$

for $k \ge 0$. For N large enough, we can use the above estimate to argue that this integral is well-defined, the sum

$$p_t(x,y) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty}(-1)^k q_t^{N,k}(x,y) \tag{14}$$

converges, and is a heat kernel for H.

The ***Hilbert–Schmidt norm*** of an operator A acting on a Hilbert space with orthonormal basis $\{e_i\}$ is defined as

$$\|A\|_{HS}^2 = \sum_{i,j}(Ae_i, e_j). \tag{15}$$

An operator A is called Hilbert–Schmidt if its Hilbert–Schmidt norm is finite. An operator is ***trace-class*** if it has the form AB where A, B are Hilbert–Schmidt. For such an operator the sum

$$\operatorname{Tr}(AB) \stackrel{\text{def}}{=} \sum_i (ABe_i, e_i), \tag{16}$$

is finite.

Let M be a compact manifold and E a Hermitian vector bundle on M. Given two sections s, s' of $E \otimes \mathrm{Dens}^{1/2}$) then $(s,s')_E = \operatorname{Tr}(s^*s')$ is a section of Dens. Denote

$$\Gamma_{L^2}(M, E \otimes \mathrm{Dens}^{1/2}) \tag{17}$$

the Hilbert space of space of square-integrable sections of $E \otimes \mathrm{Dens}^{1/2}$. If A is an operator acting on sections of $E \otimes \mathrm{Dens}^{1/2}$ with square-integrable kernel

$$\langle x|A|y\rangle \in \Gamma_{L^2}(M \times M, E \otimes \mathrm{Dens}^{1/2} \boxtimes E \otimes \mathrm{Dens}^{1/2}), \tag{18}$$

then A is trace class with

$$\operatorname{Tr}(A) = \int\limits_{x \in M} \operatorname{Tr}(\langle x|A|x\rangle). \tag{19}$$

Here, $\operatorname{Tr}(\langle x|A|x\rangle)$ is the density obtained by restricting $\langle x|A|y\rangle$ to the diagonal and applying the inner product.

If H is a generalized Laplacian acting sections of $E \otimes \mathrm{Dens}^{1/2}$, then the operator P_t associated to the heat kernel $p_t(x,y)$ of H is trace class for any $t > 0$ with trace

$$\mathrm{Tr}(P_t) = \int_{x \in M} \mathrm{Tr}(p_t(x,x)). \tag{20}$$

If E is a Hermitian vector bundle, then a generalized Laplacian H acting on sections of $E \otimes \mathrm{Dens}^{1/2}$ is symmetric if $H = H^*$, the formal adjoint of H. In this case, the operator P_t associated to the heat kernel $p_t(x,y)$ of H is equal to e^{-tH}.[1] Let $P_{(0,\infty)}$ be the projection onto the space of eigensections of H with positive eigenvalue. Then, the kernel $\langle x|P_{(0,\infty)}e^{-tH}P_{(0,\infty)}|y\rangle$ satisfies the following important bound: for t sufficiently large one has

$$\|\langle x|P_{(0,\infty)}e^{-tH}P_{(0,\infty)}|y\rangle\|_\ell \leq C(\ell)e^{-t\lambda_1} \tag{21}$$

where λ_1 is the smallest non-zero positive eigenvalue of H.

[1] More precisely, it is the closure $\overline{H}$ of H acting on $\Gamma(M, E \otimes \mathrm{Dens}^{1/2})$ that should appear here, but we will not distinguish these two operators in what follows.