# February 20

Recall, on flat space

$$q_t(x,y) = \frac{1}{(4\pi t)^{n/2}} e^{-\|x-y\|^2/4t}$$

solves the heat eqn

$$(\Delta_x + \partial_t)\, q_t(x,y) = 0 .$$

Our goal is to approximate the heat kernel on arbitrary Riem. manifold.

Toy model: $V$ = vector space

$$H \in \mathrm{End}(V).$$

Want: $P_t = e^{-tH}$

Spse we are given

$$K_t : \mathbb{R}_+ \longrightarrow \mathrm{End}(V)$$

s.t.

$$R_t \overset{dfn}{=} (\partial_t + H) K_t = \mathcal{O}(t^\alpha)$$

for some $\alpha \geq 0$, and $K_0 = \mathbb{1}$.

$K_t$ is a "parametrix" and

$R_t$ is a "remainder".

Let $\Delta^k \subset \mathbb{R}^k$ "k-simplex"

$$\Delta^k = \{ (t_1 \dots t_k) \mid 0 \leq t_1 \leq \dots \leq t_k \leq 1 \}$$

"Center of mass" coordinates

$$\sigma_0 = t_1,\ \sigma_i = t_{i+1} - t_i,\ \dots,\ \sigma_k = 1 - t_k$$

satisfying: $\sigma_0 + \cdots + \sigma_k = 1$.

For $T > 0$ write

$$T\Delta^k = \{ (t_1, \dots t_k) \mid 0 \leq t_1 \leq \cdots \leq t_k \leq T \}.$$

Lemma: $\mathrm{vol}(\Delta^k) = \frac{1}{k!}$

Pf: Induction. $\mathrm{vol}(\Delta^0) = 1$ and:

$$\mathrm{vol}(\Delta_k) = \int_0^1 \mathrm{vol}(t_k \Delta^{k-1})\, dt_k$$

$$= \int_0^1 t_k^{k-1} \mathrm{vol}(\Delta^{k-1})\, dt_k$$

$$= \frac{\mathrm{vol}(\Delta^{k-1})}{k}$$

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Thm: let $Q_t^k : \mathbb{R}_+ \longrightarrow \mathrm{End}(V)$ be

$$Q_t^k = \int_{t\Delta_k} K_{t-t_k} R_{t_k - t_{k-1}} \cdots R_{t_2 - t_1} R_{t_1}\, dt_1 \cdots dt_k.$$

$$Q_t^0 = K_t$$

Thm:

$$P_t = \sum_{k \geq 0} (-1)^k Q_t^k$$

and $P_t = K_t + \mathcal{O}(t^{1+\alpha})$.

Pf: Sps $a, b$ are $\mathbb{R}_+ \to \mathrm{End}(V)$. Then

$$\frac{\partial}{\partial t} \int_0^t a(t-s)\, b(s)\, ds$$

$$= \int_0^t \frac{\partial a}{\partial t}(t-s)\, b(s)\, ds + a(0)\, b(t).$$

Apply to $a(s) = K_s$, $b(s) = R^{(k)}(s)$,

$$R^{(k)}(s) = \int_{s\Delta^{k-1}} R_{s-t_{k-1}} \cdots R_{t_2 - t_1} R_{t_1}\, dt_1 \cdots dt_{k-1}.$$

$\Rightarrow$

$$(\partial_t + H)\, Q_t^k = R^{(k+1)}(t) + R^{(k)}(t).$$

$\rightsquigarrow$ Telescoping sum

$$(\partial_t + H) \sum_{k \geq 0} (-1)^k Q_t^k = 0 .$$

We now want to estimate $P_t - K_t$.

$$|K_{t-t_k}| \leq C_0$$

$$|R_{t_{i+1}-t_i}| \leq C t^{\alpha}$$

$\rightsquigarrow$
$$|Q_t^k| = \left| \int_{t\Delta^k} K_{t-t_k} R_{t_k - t_{k-1}} \cdots R_{t_2 - t_1} R_{t_1} \right|$$

$$\leq C_0 C^k t^{k\alpha} \underbrace{\frac{t^k}{k!}}_{\mathrm{vol}(t\Delta^k)}$$

$\Rightarrow$ $P_t = \sum_{k \geq 0} (-1)^k Q_t^k$ converges

and $P_t = K_t + O(t^{1+\alpha})$. $\boxed{\phantom{x}}$

- Now, we want to take

$$V = \Gamma(M, E \otimes \mathrm{Dens}^{1/2})$$

$$H = \Delta^{E \otimes \mathrm{Dens}^{1/2}} + F \quad \text{gen. laplacian.}$$

(1) First, we construct approximate kernel $k_t(x,y)$ and study

$$r_t(x,y) \overset{dfn}{=} (\partial_t + H_x) k_t(x,y).$$

(2) We then prove

$$\sum_{k \geq 0} (-1)^k \int_{t\Delta^k} \int_{M^k} k_{t-t_k}(x, z_k)\, r_{t_k - t_{k-1}}(z_k, z_{k-1}) \cdots r_{t_1}(z_1, y)$$

$$\xrightarrow{\text{converges}} p_t(x,y)$$

$\uparrow$ Heat kernel.

We will prove the following next time:

<u>Thm</u>: For any $N > 0$, $\exists$ smooth one parameter family of smooth $k_t^N(x, y)$ s.t. $\forall$ integers $\ell$:

① $\forall\ T > 0$, the operators $K_t^N$ form a uniformly bounded $t$-family of operators acting on

$$\Gamma^\ell(M, E \otimes \mathrm{Dens}^{1/2})$$

$\uparrow \|\cdot\|_\ell$ norm.

for $0 < t < T$.

② $\forall\ s \in \Gamma^\ell(M, E \otimes \mathrm{Dens}^{1/2})$

$$\lim_{t \to 0} K_t^N s = s \quad \text{wrt } \|\cdot\|_\ell .$$

③ $r_t^N(x, y) \overset{\text{dfn}}{=} (\partial_t + H_x)\, k_t(x, y)$ satisfies

$$\|r_t^N\|_\ell \le C(\ell)\, t^{(N - n/2) - \ell/2} .$$

Fix $N$ and write $k_t$ for $k_t^N$. Define

$Q_t^k$ to be the operator associated to

$$q_t^k(x,y) = \int_{t\Delta^k} \int_{M^k} k_{t-t_k}(x, z_k)\, r_{t_k - t_{k-1}}(z_k, z_{k-1}) \cdots r_{t_1}(z_1, y).$$

Lem: $Q_t^k = \int_{t\Delta^k} K_{t-t_k} R_{t_k - t_{k-1}} \cdots R_{t_1}$.

We need to show this is well-defined, namely that the integral converges.

Let

$$r_t^{k+1}(x,y) = \int_{t\Delta^k} \int_{M^k} r_{t-t_k}(x, z_k)\, r_{t_k - t_{k-1}}(z_k, z_{k-1}) \cdots r_{t_1}(z_1, y).$$

Lem: If $N > \frac{n+\ell}{2}$ then $r_t^{k+1}$ is $C^\ell$ wrt $x, y$ and

$$\|r_t^{k+1}\|_\ell \leq C^{k+1}\, t^{(k+1)((N-n/2)-\ell/2)} \times \mathrm{vol}(M)^k \frac{t^k}{k!}.$$

Pf: If $N > \frac{n+\ell}{2}$ then $r_t(x,y)$ and its derivatives up to order $\ell$ extend continuously to $t=0$ $\rightsquigarrow$ $r_t^{k+1}$ is well-defined. Then use:

$$\left| \int_{t\Delta^k} \int_{M^k} r_{t-t_k}(x, z_k) \cdots r_{t_1}(z_1, y) \right| \leq C^{k+1}\, t^{(k+1)((N-n/2)-\ell/2)} \cdot \mathrm{vol}(M^k) \cdot \mathrm{vol}(t\Delta^k).$$

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