January 23

- <u>Overview</u>. Let $E = E^+ \oplus E^-$ be a $\mathbb{Z}/2$ graded vector bundle on a compact manifold M. We will study special operators:

$$D : \mathcal{E} \longrightarrow \mathcal{E} = \Gamma(M, E)$$

called "generalized Dirac operators".

<u>Ex</u>: ① $E = \Lambda^\bullet T^*_M = \Lambda^{even} T^*_M \oplus \Lambda^{odd} T^*_M$.

w/ M a Riemannian manifold. Then

$$D = d + d^*$$

is an example.

② X = cplx manifold, $\begin{matrix} V \\ \downarrow \\ X \end{matrix}$ a holomorphic vector bundle w/ hermitian inner product.

Then $E = \Lambda^{\cdot} T_M^{0,1*} \otimes V$ and

Hodge adjoint

$$D = \sqrt{2}\left(\bar{\partial} + \bar{\partial}^*\right)$$

is a generalized Dirac operator.

(3) If M is spin manifold and $E = S$, associated spinor bundle and

$$D : \Gamma(M, S) \to \Gamma(M, S)$$

is the standard Dirac operator.

The <u>index</u> of D is defined by:

$$\text{index}(D) = \dim \ker D^+ - \dim \ker D^-,$$

where $D^\pm = D\big|_{E^\pm}$, this is the "super dimension" of $\ker D$.

Why is this well-defined?

Thm: [McKean-Singer] For any $t > 0$ consider the operator $e^{-tD^2} \circlearrowright \mathcal{E}$. Then:

$$\text{index}(D) = \int_M \text{str} \langle x | e^{-tD^2} | x \rangle \, d\text{vol}$$

where $\langle x | e^{-tD^2} | y \rangle : \mathcal{E}_y \to \mathcal{E}_x$ is the *integral kernel* of the operator e^{-tD^2}:

$$\left(e^{-tD^2} s\right)(x) = \int_M \underbrace{\langle x | e^{-tD^2} | y \rangle}_{\text{"heat kernel"}} s(y) \, d\text{vol}_y .$$

- The key feature of this result is that it holds at an arbitrary $t > 0$! The first few weeks of the class will focus on the small t asymptotic behavior of the heat kernel.

Using this, our first main result will be a proof of the Atiyah-Singer-Patodi index theorem.

chern character

$$\operatorname{index}(\mathcal{D}) = \int_M \hat{A}(M)\, \operatorname{ch}(\mathcal{E}/S).$$

$\hat{A}$-genus.

Indeed, this will follow from showing

$$\lim_{t \to 0} \langle x | e^{-t\mathcal{D}^2} | x \rangle$$

exists and is equal to the integrand above. This is known as the "local" index theorem, and is due to Patodi, Gilkey.

- The heat equation is

on $\mathbb{R}^n$

$$\partial_t u = \Delta u = \partial_{x_1}^2 u + \cdots + \partial_{x_n}^2 u .$$

We study this equation on $M \times M \times \mathbb{R}_+$

Riemannian

We provide the "initial condition":

$$\lim_{t \to 0} k_t(x,y) = \delta(x-y)$$

$$\int_{x \in M} f(x)\, \delta(x-y) = f(y)$$

Magically, the small t asymptotics know of the local geometry of M.

$$k_t(x,y) \underset{t \sim 0}{\sim} (4\pi t)^{-n/2} e^{-\|x-y\|^2/4t} \sum_{i \geq 0} t^i f_i(x,y).$$

· The $\hat{A}$-genus appearing in the index formula is a special characteristic class. We will define it soon, but here are some properties.

For M a smooth manifold, the ith Pontryagin class $p_i^{(M)} \in H^{4i}(M)$ is

complexification

$$p_i(M) = (-1)^i c_{2i}(T_M \otimes \mathbb{C}).$$

Then

$$\hat{A}(M) = 1 + \hat{A}_1(M) + \hat{A}_2(M) + \cdots$$

w/

$$\hat{A}_1(M) = -\frac{1}{24} p_1(M)$$

$$\hat{A}_2(M) = \frac{1}{5760}\left(-4 p_2(M) + 7 p_1(M)^2\right)$$

$$\hat{A}_3(M) = \frac{1}{967680}\left(-16 p_3(M) + 44 p_1(M) p_2(M) - 31 p_1(M)^3\right)$$

Various enhancements / applications:

① An equivariant index theorem, here

$G \circlearrowright M$ and $G \circlearrowright E$.

② A "family" index theorem. Here, the bundle E (and hence the Dirac operator D) become a family of bundles parametrized by some manifold B.

$$\begin{tikzcd} \pi^{-1}(z) \arrow[d] & \subset & M \arrow[d, "\pi"] & \rightsquigarrow & \{D^z\} \\ z & \longrightarrow & B & & \end{tikzcd}$$

family of Dirac operators.

This is originally due to Bismut.

③ Applications in topology, physics, etc...

Anomalies.