January 30

Differential operators.

Let M be a manifold and $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$ a vector bundle.

$$\mathrm{End}\left(\Gamma(M, E)\right)$$

is a huge algebra. Let $\mathcal{D}(M, E)$ be the subalgebra generated by

1) $\Gamma(M, \mathrm{End}\, E)$, acting by multiplication.

2) Covariant derivatives ∇_X, where ∇ is any connection and X ranges over all vector fields on M.

The algebra $\mathcal{D}(M, E)$ is equipped w/ a filtration:

$$F^i \mathcal{D}(M,E) = \Gamma(M, \mathrm{End}\,E) \cdot \mathrm{span}\{\nabla_{x_1} \cdots \nabla_{x_j} \mid j \leq i\}.$$

"ith order differential operators".

- If $A = \bigcup_i F^i A$ is an filtered algebra

$$F^i \subset F^{i+1}$$

$$F^i \cdot F^j \subset F^{i+j}$$

then the associated graded is the algebra:

$$\mathrm{gr}\,A \overset{\mathrm{dfn}}{=} \bigoplus_i \underbrace{F^i / F^{i-1}}_{\mathrm{gr}^i A}$$

$$\begin{tikzcd} \mathrm{gr}\,A \arrow[r, phantom, "="] & \bigoplus_i F^i/F^{i-1} \\ A \arrow[u, "\sigma"] \arrow[r, phantom, "\supset"] & F^i \arrow[u] \end{tikzcd}$$

symbol

<u>Prop</u>: $\mathrm{gr}\, \mathcal{D}(M,E) \xrightarrow{\cong} \Gamma\left(M, S^{\cdot}(T_M) \otimes \mathrm{End}\, E\right)$.

where the isomorphism:

$$\sigma_k : \mathrm{gr}^k \mathcal{D}(M,E) \xrightarrow{\cong} \Gamma\left(M, S^k(T_M) \otimes \mathrm{End}\, E\right).$$

is given by

$$\sigma_k(D)(x,\xi) = \lim_{t \to \infty} t^{-k} \left(e^{-itf} \cdot D \cdot e^{itf} \right)(x) \in \mathrm{End}(E_x).$$

where $D \in \mathcal{D}_k(M,E)$, $x \in M$, $\xi \in T_x^* M$, and f is any smooth fn s.t. $df(x) = \xi$.

<u>Pf</u>: Let $a \in \Gamma(M, \mathrm{End}\, E)$ and consider

$$D = a \nabla_{X_1} \cdots \nabla_{X_k}$$

By Leibniz rule

leading order in t.

$$e^{-itf} D e^{itf} = (it)^k a(x) \langle X_1(x), \xi \rangle \cdots \langle X_k(x), \xi \rangle + O(t^{k-1})$$

$\Rightarrow \lim_{t \to \infty} t^{-k} (-)$ is independent of f,

and defines the line isomorphism. $\boxtimes$

We can identify

$$\Gamma(M, S(T_M) \otimes \operatorname{End} E) \subset \Gamma(T^*M, \pi^* \operatorname{End} E)$$

w/ sections that are polynomial along the fibers of $\pi : T^*M \to M$.

Dfn: We say a differential operator D of order k is *elliptic* if $\sigma_k(D)$ is *invertible* over the open set

$$\{(x, \xi) \mid x \in M,\ \xi \neq 0\} \subset T^* M .$$

Ex: · A *zeroth* order differential operator is simply an endomorphism of E. It is elliptic $\Leftrightarrow$ it is invertible.

· Any *first-order* differential operator is of the form

$$\sigma_1 \circ \nabla + \sigma_0$$

where
- ∇ is a connection on E.
- $\sigma_i \in \Gamma(M, S^i T_M \otimes \operatorname{End} E)$.

Here, the first term means

$$\begin{tikzcd}
\Gamma(M, E) \arrow[r, "\nabla"] \arrow[dr, dashed, "\sigma_1 \circ \nabla"'] & \Gamma(M, T^*_M \otimes E) \arrow[d, "\sigma_1"] \\
& \Gamma(M, E)
\end{tikzcd}$$

The symbol of this differential operator is σ_1. When is this operator elliptic?

<u>Dfn</u>: Let $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$ be a v.b. over (M, g) a Riemannian manifold. A <u>generalized Laplacian</u> on E is a second order differential operator H s.t.

$$\sigma_2(H)(x, \xi) = |\xi|^2 .$$

Clearly any generalized Laplacian is elliptic. In local coordinates:

$$H = -\sum_{i,j} \underbrace{g^{ij}(x)}_{\text{metric on } T^*_M} \partial_i \partial_j + (\text{first-order})$$

Prop (HW problem): H is generalized Laplacian

$\iff$ $[[H, f], g] = -2 \langle \operatorname{grad} f, \operatorname{grad} g \rangle$

$\forall f \in C^\infty(M)$.

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E
$\downarrow$ v.b. on Riemannian manifold.
M

∇ = Levi-Civita connection

∇^E = connection on E.

$\rightsquigarrow$ $\nabla \otimes \nabla^E$ connection on $T^*_M \otimes E$.

$\rightsquigarrow$

$$\begin{tikzcd}
\Gamma(M,E) \arrow[r, "\nabla^E"] \arrow[dr, dashed, "D^E"'] & \Gamma(M, T^*_M \otimes E) \arrow[d, "\nabla \otimes \nabla^E"] & \\
& \Gamma(M, T^*_M \otimes T^*_M \otimes E) \arrow[dr, "-\mathrm{Tr}"] & \\
& & \Gamma(M,E)
\end{tikzcd}$$

Define the Laplacian:

$$\Delta^E : \Gamma(M,E) \longrightarrow \Gamma(M,E)$$

by
$$\boxed{\Delta^E s = -\mathrm{Tr}\left(D^E s\right)}$$

- Explicitly, for $X, Y \in \mathrm{Vect}(M)$

$$\left(D^E s\right)(X,Y) = \left(\nabla^E_X \nabla^E_Y - \nabla^E_{\nabla_X Y}\right) s$$