Notes for MA 822: Topics in geometry

Hilbert schemes, quiver varieties, and infinite-dimensional algebras

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LECTURE 1

Introduction to the class

Popularized and pioneered by Grothendieck, Hilbert schemes are among the most fundamental moduli spaces in algebraic geometry. Given an algebraic variety one can study the space which parameterizes all possible subschemes of the fixed variety; this space is called a Hilbert scheme. Throughout this course we will study the Hilbert scheme of dimension zero subvarieties (points) of smooth algebraic surfaces. There is a nice balance of richness and accessibility in the context of Hilbert schemes of points in algebraic surfaces. On one hand, the Hilbert scheme of points of algebraic curves agrees with the symmetric product of the curve; so this is a rather trivial case of Hilbert schemes. On the other hand, for smooth algebraic varieties of dimension at least three, the Hilbert scheme of points is generally singular. It is in the case of dimension zero subschemes of a smooth algebraic variety where the Hilbert scheme is smooth and irreducible.

In the case that the algebraic surface is affine space $\mathbf{A}^2$, one think about Hilbert schemes as a particular moduli space of rank one torsion-free sheaves on the projective variety $\mathbb{P}^2$. More generally one can consider moduli spaces of torsion-free sheaves of higher rank. These moduli spaces are amenable to similar tools and techniques that one uses for Hilbert schemes.

Part of the goal of this course is to elucidate algebro-geometric properties of Hilbert schemes of points and moduli spaces of torsion-free sheaves on smooth algebraic surfaces. For the lecturer, however, perhaps most fascinating is the connection between these moduli spaces and at least three other topics:

1. Moduli spaces of instantons on $\mathbf{R}^4$ and more generally singularities of ALE type.
2. Representation theory of infinite-dimensional Lie algebras such as affine Kac–Moody algebras.
3. String theory and M theory. Specifically the infamous theory $\mathfrak{X}$ which one can think about as a twist of the worldvolume theory of fivebranes in M theory.

We will mostly be following the books [Nak99; Kir16].

1.1. Quiver representations and instantons

Solutions to the anti-self-dual Yang–Mills equations on a four-dimensional manifold M are called *instantons*. Amazingly, by work of Atiyah, Drinfeld, Hitchin, and Manin (ADHM) for $M = \mathbf{R}^4$ gauge equivalent classes of such connections can be described in terms of solutions of some quadratic equations for certain finite-dimensional matrices [Ati+78]. This relies on a presentation of $\mathbf{R}^4 = \mathbf{C}^2$ as a hyper Kähler space. Let us briefly sketch this approach for $U(N)$ instantons of 'charge k'. Here, charge is given by $\int_{S^4} F_A \wedge F_A$ normalized so that it is an integer.

Fix the following data:

- a pair of complex vector spaces V, W of dimensions k, N, respectively.
- a pair of complex endomorphisms $X, Y \colon V \to V$.
- a pair of linear maps $i \colon W \to V$ and $j \colon V \to W$.

These are required to satisfy the ADHM equation

$$XY - YX + ij = 0, \tag{1}$$

together with a non-degeneracy (or stability) condition. From this data one constructs an instanton on $\mathbf{R}^4$ which has rank N and topological charge k. This data can be extracted from the so-called "ADHM quiver", see figure **??**. More precisely, this data determines a representation of the ADHM quiver—roughly, the vector spaces V, W label the nodes (there is a framed and an unframed node) and the morphisms are labeled by the edges. The word 'quiver' simply refers to a directed graph. The maneuver of associating to a quiver the above linear data will be explained in this course.

To connect to the Hilbert scheme on $\mathbf{A}^2$ one should look at rank one instantons on $\mathbf{R}^4$. Strictly speaking, there are no instantons, but a slight variant of the above construction in terms of the ADHM quiver returns the Hilbert scheme. Roughly speaking, the moduli space of 'non-commutative' rank one instantons of charge k can be identified with the Hilbert scheme of k points on $\mathbf{A}^2$.

On a more general class of non-compact four-manifolds which are *asymptotically locally Euclidean* (ALE) there is a description of instantons in terms of more general quadratic equations also defined on some space of finite-dimensional matrices [KN90; Nak94]. The corresponding moduli spaces can be described in terms of a more general class of quivers whose underlying graphs are the Dynkin graphs of type ADE. This is not an accident: there is a classification due to Kronheimer of four-dimensional ALE spaces: they resolutions of singularities of the form $\mathbf{C}^2/\Gamma$ where $\Gamma \subset SU(2)$ is a finite subgroup. Finite subgroups of $SU(2)$ fall under the same ADE classification as finite simple Lie groups. For this reason, sometimes $\mathbf{C}^2/\Gamma$ is referred to as a ADE singularity. The relationship between these related classifications follows from the fact that both structures are governed by the combinatorics of the simply laced Dynkin diagrams.

Even if we forget the gauge theoretic origin, associated to any quiver is a moduli space of representations called the *Nakajima quiver variety*. These will be the main geometric objects we are concerned with in this course. In many cases, there are independent, algebro-geometric descriptions of these moduli spaces in terms of sheaves on complex varieties of dimension two. For example, in the ADHM case, it corresponds to the moduli space of torsion-free sheaves on $\mathbb{P}^2$ which are of rank N, framed at $\infty \in \mathbb{P}^2$, and have second Chern class equal to k.

1.2. Infinite-dimensional Lie algebras

Cohomology is one of the fundamental invariants of a space. In this course we will give a description of the cohomology of the Hilbert scheme and of more general quiver varieties. For the Hilbert scheme we will work out a beautiful formula for the generating function of the Poincaré polynomial derived originally by Göttsche [Göt90]. This generating function is of the form

$$\sum_{n\geq 0} q^n P_{\mathrm{Hilb}_n(X)}(t). \tag{2}$$

Here $P_Y(t) = \sum_n t^n \cdot (\dim H^n(Y))$ is the Poincaré polynomial of a space Y.

Amazingly, this generating function matches with an expression for the character of a representations for a certain infinite-dimensional algebra called the Heisenberg algebra. The Heisenberg algebra $\mathfrak{heis}$ is a central extension of the abelian Lie algebra of Laurent polynomials in a single variable

$$\mathbf{C} \to \mathfrak{heis} \to \mathbf{C}((z)). \tag{3}$$

It has irreducible representations labeled by a 'level' and a highest weight. One of the main results of Nakajima [Nak97] and Grojnowski [Gro96] is that for X an algebraic surface the direct sum of the homologies of Hilbert schemes

$$\oplus_{n\geq 0} H_\bullet\left(\mathrm{Hilb}_n(X)\right) \tag{4}$$

is a representation for $\mathfrak{heis}$. Moreover, the action of the Heisenberg algebra can be constructed in a completely geometric way and each $H_\bullet(\mathrm{Hilb}_n(X))$ is a weight space. Furthermore, one can actually argue that a richer structure is present on the direct sum above.

A *vertex algebra* is a structure present in two-dimensional conformal field theory (CFT). It is the algebraic structure carried by the so-called 'local operators' of a holomorphic two-dimensional CFT. The direct sum above turns out to be a vertex algebra in a totally geometric way.

For the case of higher ranks or more general quivers there is a similar picture. Here, the Heisenberg algebra is replaced by a Lie algebras of affine Kac–Moody type [Kac90].

1.3. Connection to string theory

A natural question to ask is for an 'explanation' for the relationship between CFT and the Hilbert scheme or instanton moduli spaces. One potential answer can be found in *string theory*. The Hilbert scheme of points and its higher rank generalizations take part in a rich collection of dualities in string theory such as the correspondence of Alday, Gaiotto, and Tachikawa [AGT10].

There is an infamous six-dimensional supersymmetric quantum field theory which can be defined for any Lie algebra of type ADE, just as in the classification of ALE spaces. From the point of view of string theory, one can think about the theory as obtained from 'compactifying' string theory on an ALE space. Though, no rigorous description of this theory exists, physicists are still able to glean useful information from this setup.

A setup relevant to the above discussion is to consider the six-dimensional theory on a product of manifolds of the form

$$\Sigma \times X \tag{5}$$

where Σ is a Riemann surface and X is a complex two-dimensional surface. The 'compactification' in the X direction yields a two-dimensional CFT whose states bear a close relationship to the cohomology of moduli spaces we will be considering. The remaining Σ-direction exhibits the rich structure of a CFT that we alluded to above.

LECTURE 2

The Hilbert scheme of points on a surface

In this lecture we introduce symmetric products of algebraic varieties, give a scheme-theoretic definition of the Hilbert scheme of points, and introduce an explicit description using geometric invariant theory. The justification of this description is where we will go in the next few lectures. Much of what we do in this course works over an arbitrary algebraically closed field. For the most part we will restrict ourselves to working over $\mathbf{C}$.

2.1. Symmetric products and Hilbert schemes

Let's begin with a simple example. Given any topological space X we can consider the n-fold symmetric product

$$S^n X = X^{\times n}/S_n \tag{6}$$

where the symmetric group S_n acts on the cartesian product $X^{\times n}$ in the natural way. Notice that if X is a smooth manifold it is no longer the case that $S^n X$ is a smooth manifold. The problem is that there are singular points (so-called 'orbifold' points.) In a sense, the Hilbert scheme of points on X is a 'resolution' of these singularities.

There is the following algebraic interpretation of the symmetric product. For example, suppose that X is just (complex) algebraic affine space $\mathbf{A}^1 = \mathrm{Spec}(\mathbf{C}[x])$. Then, we have the following presentation

$$S^n \mathbf{A}^1 = \mathrm{Spec}\left(\mathbf{C}[x_1, x_2, \ldots, x_n]^{S_n}\right) \tag{7}$$

where S_n permutes the variables x_i in the defining way. By classical invariant theory one knows that

$$\mathbf{C}[x_1, \ldots, x_n]^{S_n} \simeq \mathbf{C}[s_1, \ldots, s_n] \tag{8}$$

where s_n are the elementary symmetric polynomials in n-variables. Thus $S^n\mathbf{A}^1 \simeq \mathbf{A}^n$ as algebraic varieties.

More generally, we have the following definition of the symmetric power of an arbitrary affine algebraic variety X as

$$S^n X \stackrel{\text{def}}{=} \mathrm{Spec}\left(\left(\mathbf{C}[X]^{\otimes n}\right)^{S_n}\right). \tag{9}$$

That is, the spectrum of the S_n invariants of the ring $\mathbf{C}[X]^{\otimes n}$, where $\mathbf{C}[X]$ is the ring of regular functions on X.

In higher dimensions, the symmetric powers of a smooth variety can be singular. Take for example the affine algebraic variety $X = \mathbf{A}^2 = \mathrm{Spec}\,\mathbf{C}[z_1, z_2]$. By definition, the symmetric square $S^2\mathbf{A}^2$ is the spectrum of the following ring

$$A = \mathbf{C}[z_1, z_2, w_1, w_2]^{\mathbf{Z}/2} \tag{10}$$

where $\mathbf{Z}/2$ acts by $z_i \leftrightarrow w_i$ for $i = 1, 2$.

Proposition 2.1.1. *There is an isomorphism of rings*

$$A \simeq \mathbf{C}[x, y, u, v, w]/(uv - w^2). \tag{11}$$

In particular

$$S^2\mathbf{A}^2 \simeq \mathbf{A}^2 \times Q \tag{12}$$

where Q is the (singular) quadric in $\mathbf{C}^3$ defined by $uv = w^2$.

PROOF. Make the following change of variables $x = z_1 + w_1, y = z_2 + w_2, s = z_1 - w_1, t = z_2 - w_2$. Then, the $\mathbf{Z}/2$ action on these new variables leaves x, y invariant so that

$$A \simeq \mathbf{C}[x, y] \otimes B \tag{13}$$

where $B = \mathbf{C}[s, t]^{\mathbf{Z}/2}$ and the new $\mathbf{Z}/2$ action is $s \to -s$, $t \to -t$. If we further reparameterize $u = s^2$, $v = t^2$, $w = st$ we see that

$$B \simeq \mathbf{C}[u, v, w]/(uv - w^2). \tag{14}$$

□

We want to do better than the symmetric product. Let $X = \mathrm{Spec}(A)$ be an affine algebraic variety. The ***Hilbert scheme*** of n-points in X has underlying set defined by

$$\mathrm{Hilb}_n(X) \stackrel{\mathrm{def}}{=} \{J \subset A \mid J \,\text{ideal}, \dim(A/J) = n\}. \tag{15}$$

When $\dim X = 1$ it is easy to see that $\mathrm{Hilb}_n(X) = S^n X$. But more generally, the Hilbert schemes differ from the symmetric powers. There is, however, a natural map

$$\pi_{HC} \colon \mathrm{Hilb}_n(X) \to S^n X \tag{16}$$

called the ***Hilbert–Chow*** morphism. On sets, it sends an ideal $J \subset A$ to the support $\mathrm{supp}(A/J)$.

Remark 2.1.2. Here, if M is an A-module then its support $\mathrm{supp}(M) \subset X = \mathrm{Spec}(A)$ can be thought of as an unordered set of points in X.

Remark 2.1.3. If $\dim X = 2$, and X is nonsingular, then the Hilbert–Chow morphism is a resolution of singularities.

This is the definition of the Hilbert scheme as a set. Below we will see how one endows it with the structure of a scheme.

Here is another useful presentation of the Hilbert scheme as a set.

Lemma 2.1.4. *Let $X = \mathrm{Spec}(A)$ be an affine variety. There is a bijection of $\mathrm{Hilb}_n(X)$ with the set of pairs*

$$(M, v) \tag{17}$$

*where M is an A-module of dimension n and $v \in M$ is a vector which satisfies $A \cdot v = M$ (such a vector is called a **cyclic** vector).*

PROOF. In one direction the correspondence takes an ideal J and sends it to $M = \mathbf{C}[X]/J$ with $v = 1$. $\square$

2.2. REPRESENTABILITY

In the next section we will define the Hilbert scheme using the functor of points perspective. The main objects are functors of the form

$$F\colon (\mathrm{Sch}_{/S})^{op} \to \mathrm{Sets} \tag{18}$$

where $\mathrm{Sch}_{/S}$ is the category of schemes over a fixed scheme S and the op denotes the opposite category.

An important example of a contravariant functor is the following. Suppose that X is any scheme. Then consider the "Yoneda" functor

$$h_{X/S}\colon (\mathrm{Sch}_{/S})^{op} \to \mathrm{Sets} \tag{19}$$

defined by

$$h_{X/S}(U) = \mathrm{Hom}_{\mathrm{Sch}_{/S}}(U, X). \tag{20}$$

When we write U on the left hand side we implicitly remember it is a scheme over S, that is, it comes with a morphism $U \to S$.

Given a functor $F\colon (\mathrm{Sch}_{/S})^{op} \to \mathrm{Sets}$ we want to know whether it is representable. This means that there is an equivalence of functors $h_X \simeq F$ for some scheme X. In this case, we then say that X represents F. Let's unpack what such an equivalence would mean.

First off, an equivalence of functors means we have a natural transformation $\eta\colon h_X \to F$. There is a canonical element in $h_X(X)$ given by the identity $\mathbb{1}_Y$. Via the transformation η we obtain an element

$$\xi \stackrel{\mathrm{def}}{=} \eta(\mathbb{1}_X) \in F(X). \tag{21}$$

Conversely, given an element $\xi \in F(X)$ we can construct a natural transformation $\eta_\xi\colon h_X \to F$ as follows. For any $f\colon Y \to X$ in $h_X(Y)$ let $\eta_\xi(f) = f^*\xi$. (Here $f^*\xi$ stands for the image of ξ under the map $F(f)\colon F(Y) \to F(X)$.) One can see that these two operations are inverses to one another which gives the "Yoneda lemma"

$$\mathrm{Fun}(h_X, F) \simeq F(X). \tag{22}$$

Thus we can rephrase representability as follows.

Definition 2.2.1. Let F be a functor as above. A pair (X, ξ) where X is a scheme over S and $\xi \in F(X)$ ***represents*** F if the induced natural transformation $\eta_\xi\colon h_{X/S} \to F$ is an equivalence. Equivalently, for any $T \to S$ there is a natural one-to-one correspondence between lifts

$$\begin{array}{ccc} & & X \\ & \overset{\phi}{\nearrow} & \downarrow \\ T & \longrightarrow & S. \end{array} \tag{23}$$

and elements $\phi^*\xi \in F(T)$.

The element ξ is usually called the *universal family* corresponding to F.

Example 2.2.2 (Point functor for projective space)**.** Let $S = \operatorname{Spec} \mathbf{C}$ for concreteness. Which functor represents projective space $\mathbb{P}^n$? Recall that $\mathbb{P}^n$ is the space of lines in $\mathbf{C}^{n+1}$. Thus, for a scheme X we can think about a map

$$\phi\colon X \to \mathbb{P}^n \tag{24}$$

as giving a family of lines in $\mathbf{C}^{n+1}$ parametrized by X. Conversely, given a family of lines in $\mathbf{C}^{n+1}$ parametrized by X we should be able to construct such a map.

The important question is what does 'family' of lines mean in this context? A first attempt would be to define a family of lines in $\mathbf{C}^{n+1}$ parametrized by X as a sub vector bundle of the trivial rank $n+1$ bundle $\mathcal{O}_X^{\oplus n+1}$. The problem with this is that sub bundles are not so well-behaved sheaf-theoretically. Indeed, if $\mathcal{F}$ is a locally free sheaf with corresponding bundle F and $\mathcal{E} \subset \mathcal{F}$ is a locally free subsheaf with corresponding bundle E, then the map on stalks $E_x \to F_x$ may not be injective. Better, then, is to look at locally free subsheaves.

For a fixed scheme X let $F(X)$ be the set of exact sequences

$$0 \to \mathcal{K} \to \mathcal{O}_X^{\oplus n+1} \to \mathcal{L} \to 0 \tag{25}$$

up to equivalence where $\mathcal{L}$ (or $\mathcal{K}$) is rank one. We can upgrade $X \mapsto F(X)$ to a functor as above; indeed, pulling back sheaves along $X \to Y$ results in a map $F(Y) \to F(X)$.

The functor F is represented by the projective space in the sense that there is a natural bijective correspodence between maps $\phi\colon X \to \mathbb{P}^n$ and elements of $F(X)$.

2.3. Hilbert schemes: formal definition

So far we have only provided the careful definition of the Hilbert scheme for affine varieties. Even in this case we didn't give an argument as to why it has the structure of a scheme. The goal of this section is to remedy these two shortcomings using the functor of points perspective. We will state, but not prove, a very important result that the Hilbert scheme functor is representatable in an extremely general situation. Later on, for Hilbert schemes on $\mathbf{A}^2$ we will come up with an explicit presentation.

First a definition.

Definition 2.3.1. Let X be a scheme over S. An ***algebraic family of closed subschemes*** of X/S parameterized by a scheme T is a closed subscheme

$$Z \subset X_T \stackrel{\text{def}}{=} X \times_S T. \tag{26}$$

The family is ***flat*** if the induced morphism $Z \to X_T \to T$ is flat.

For the most part we will take $S = \operatorname{Spec} \mathbf{C}$. Fix a projective scheme X over $\operatorname{Spec} \mathbf{C}$ and let $\mathrm{Sch}_{/\mathbf{C}}$ be the category of all schemes over $\operatorname{Spec} \mathbf{C}$. Define the functor

$$\mathcal{H}\mathrm{ilb}_X\colon (\mathrm{Sch}_{/\mathbf{C}})^{op} \to \mathrm{Sets} \tag{27}$$

by sending a scheme T to the set of flat algebraic families of closed subschemes parametrized by T. If $f\colon T \to T'$ is a map of schemes then any T'-family $Z \subset X \times T'$

restricts along f to a T-family $f^*Z \subset X \times T'$. Thus, $\mathcal{H}\mathrm{ilb}_X$ is a contraviant functor from the category of schemes over $\mathbf{C}$ to sets.

Now, we will define a subfunctor of $\mathcal{H}\mathrm{ilb}_X$ with some nice properties. Let $\mathcal{H}\mathrm{ilb}_{X,n}$ be the subfunctor which assigns to a scheme T the set of families with Hilbert polynomial P.

Aside 2.3.2 (Hilbert polynomial)**.** The Euler characteristic of a sheaf $\mathcal{F}$ on X is

$$\chi(X, \mathcal{F}) \stackrel{\text{def}}{=} \sum_i (-1)^i \dim H^i(X, \mathcal{F}). \tag{28}$$

Let $j \colon X \hookrightarrow \mathbb{P}^N$ be a projective scheme. For $m \geq 0$ let $\mathcal{O}_X(m) = j^{-1}\mathcal{O}_{\mathbb{P}^N}(m)$ and $\mathcal{F}(m) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(m)$. The Hilbert polynomial of $\mathcal{F}$ is defined by

$$P_{\mathcal{F}}(m) \stackrel{\text{def}}{=} \chi(X, \mathcal{F}(m)). \tag{29}$$

The fact that this is actually a polynomial requires a bit of work.

If $Z \subset X \times T$ is a closed family of subschemes parametrized by T then we let

$$P_t(m) \stackrel{\text{def}}{=} P_{\mathcal{O}_{Z_t}}(m). \tag{30}$$

By flatness, when T is connected this polynomial is independent of $t \in T$. In this case we simply denoted by P.

THEOREM 2.3.3 ([Gro95]). *Let X be a projective scheme. Then, the functor $\mathcal{H}\mathrm{ilb}_{X,P}$ is representable by a projective scheme that we denote by $\mathrm{Hilb}_P(X)$. In particular, this means that there is a universal family $\mathcal{Z}_{X,P} \to \mathrm{Hilb}_P(X)$ such that every family on a scheme U is determined by restricting this family via a unique morphism $U \to \mathrm{Hilb}_P(X)$.*

Remark 2.3.4. This theorem implies that the full Hilbert scheme $\mathcal{H}\mathrm{ilb}_X$ (with no condition on the Hilbert polynomial) is represented by the scheme

$$\bigsqcup_P \mathrm{Hilb}_P(X). \tag{31}$$

This result allows us to define the Hilbert scheme for any quasi-projective scheme. Indeed, if $Y \subset X$ is an open subscheme of a projective scheme then we have the corresponding open subscheme $\mathrm{Hilb}_P(Y) \subset \mathrm{Hilb}_P(X)$.

Definition 2.3.5. Let P be the constant polynomial $P = n$. Then, we denote $\mathrm{Hilb}_P(X) = \mathrm{Hilb}_n(X)$ and call it the Hilbert scheme of n points on X.

It is worthwhile to see that in the case that X is an affine algebraic variety that this definition agrees with (15). For the most part, we will restrict ourselves to Hilbert schemes of points on schemes of dimension two. We will give an explicit descriptions of the Hilbert scheme $\mathrm{Hilb}_n(\mathbf{A}^2)$. Via a gluing argument, one can define the Hilbert scheme associated to any nonsingular complex surface in the complex analytic category [Dou66]. In particular, for X a complex analytic surface the space $\mathrm{Hilb}_n(X)$ is defined and has the structure of a complex manifold.

2.4. AN EXPLICIT DESCRIPTION

Suppose that $X = \mathrm{Spec}(A)$ is an affine algebraic variety and that G is a linear algebraic group acting algebraically on X (all defined over $\mathbf{C}$). We will also assume that G is *reductive* meaning that its radical is a torus. It is in this case that the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ is a direct sum of semisimple and commutative Lie algebras.

Definition 2.4.1. The ***geometric invariant theory (GIT) quotient*** of $X = \mathrm{Spec}\, A$ by an algebraic G-action is the affine algebraic variety

$$X \mathbin{/\!/} G \stackrel{\text{def}}{=} \mathrm{Spec}\left(A^G\right). \tag{32}$$

Remark 2.4.2. It is a theorem of Hilbert that the algebra $\mathbf{C}[X]^G$ is finitely generated. Therefore the set of maximal ideals indeed defines an affine algebraic variety.

We will study GIT quotients in more detail during the next lecture. For the time being, we will introduce an explicit GIT description of the Hilbert scheme.

Fix an integer n and consider the following (non-linear) subspace

$$H_n \subset \mathrm{Hom}(\mathbf{C}^n, \mathbf{C}^n)^{\oplus 2} \oplus \mathbf{C}^n \oplus (\mathbf{C}^n)^* \tag{33}$$

as the set of tuples (X, Y, i, j) which satisfy

$$[X, Y] - ij = 0. \tag{34}$$

Also let

$$H_n^s \subset H_n \tag{35}$$

be the subspace where i generates $\mathbf{C}^n$ under the action by X, Y. There is a natural action of $GL(n, \mathbf{C})$ on H_n and H_n^s.

The proof of the following result will occupy the next few lectures.

THEOREM 2.4.3. *There are isomorphisms of algebraic varieties*

$$S^n\mathbf{A}^2 \simeq H_n \mathbin{/\!/} GL(n, \mathbf{C})$$
$$\mathrm{Hilb}_n(\mathbf{A}^2) \simeq H_n^s \mathbin{/\!/} GL(n, \mathbf{C}).$$

Moreover, the natural map $H_n^s \hookrightarrow H_n$ induces the Hilbert–Chow morphism

$$\pi_{HC} \colon \mathrm{Hilb}_n(\mathbf{A}^2) \to S^n\mathbf{A}^2. \tag{36}$$

Remark 2.4.4. Actually, one can obtain a slightly more minimal description of $\mathrm{Hilb}_n(\mathbf{A}^2)$. The condition that i generates $\mathbf{C}^n$ under the action of X, Y together with (62) can be shown to imply that $j = 0$. Thus $\mathrm{Hilb}_n(X)$ can be realized as the $GL(n; \mathbf{C})$ quotient of the set of triples (X, Y, i) such that $[X, Y] = 0$ and that i generates $\mathbf{C}^n$ under the action by X, Y.

LECTURE 3

Geometric invariant theory, I

Studying objects up to a notion of equivalence is an integral concept in any area of mathematics. The appearance of quotients is thus inevitable. In this lecture we begin studying quotients which are particularly well-behaved in the algebro-geometric setting.

3.1. Quotients in geometry and topology

We will review some classic results about group actions on topological spaces and smooth manifolds.

If G is a group acting on a topological space M then we can consider the set-theoretic quotient M/G which is, by definition, the set of G-orbits. This set is equipped with a natural topology for which the map $M \to M/G$ is continuous. But, in general, this topological space may not even be Hausdorff. To get a nicer behaved quotient we look at a more refined situation.

Suppose that G is a real Lie group acting on a smooth manifold M. We call this action ***proper*** if the map

$$\begin{aligned} G \times M &\to M \times M \\ (g, m) &\mapsto (m, g \cdot m) \end{aligned}$$

is proper; meaning the preimage of any compact set is compact. Equivalently, this is the condition that for any compact sets $K, K' \subset M$ that the subset $\{g \in G \mid gK \cap K'\} \subset G$ is compact.

Denote the stabilizer of a point $x \in M$ by

$$G_x \stackrel{\text{def}}{=} \{g \in G \mid g \cdot x = x\} \subset G. \tag{37}$$

Also, denote the orbit of a point $x \in M$ by

$$\mathbb{O}_x \stackrel{\text{def}}{=} \{y \in M \mid y = g \cdot x\} \subset M. \tag{38}$$

We have the following easy observations:

- if G is compact then properness is guaranteed.
- any G-orbit of a proper action is a closed subset of M.

A fundamental result pertaining to quotients spaces in geometry and topology is the following so-called "slice theorem".

THEOREM 3.1.1. *Suppose G acts properly on a smooth manifold M. For every $x \in M$ there exists a locally closed submanifold $S_x \subset M$ (called a* **slice***) containing x which is* *invariant under the action of* G_x*. Furthermore, there exists an open neighborhood* $U \supset \mathbb{O}_x$ *such that the natural map*

$$G \times_{G_x} S \xrightarrow{\simeq} U \tag{39}$$

is a homeomorphism.

In particular, $\mathbb{O}_x$ *is a smooth closed submanifold of* M*.*

One can think about a slice as a closed submanifold which is transverse to the G-action. In other words, acting on S by the entire group G sweeps out the entire manifold. A local slice just sweeps out an open neighborhood of an orbit.

Example 3.1.2. Consider the action of $SO(2) = U(1)$ on $\mathbf{R}^2$ by rotations. The stabilizer at $x \neq 0$ is trivial. A slice through x is the line through x and the origin.

As a corollary one obtains the following.

THEOREM 3.1.3. *Suppose that* G *acts freely and properly on a smooth manifold* M*. Then the set of orbits* M/G *has the structure of smooth manifold with the property that the quotient map*

$$M \to M/G \tag{40}$$

is a smooth principal G*-bundle.*

So far these theorems take place in the smooth or topological setting. We now move towards quotients in algebraic geometry.

3.2. ALGEBRAIC GROUPS

One has the following hierarchy

$$\begin{aligned}\{\text{groups}\} \supset \{\text{topological groups}\} \supset \{\text{real Lie groups}\} \\ \supset \{\text{complex Lie groups}\} \supset \{\text{algebraic groups}\}\end{aligned} \tag{41}$$

and for actions $G \times X \to X$ we have

$$\begin{aligned}\{\text{action}\} \supset \{\text{continuous action}\} \supset \{\text{smooth action}\} \\ \supset \{\text{holomorphic action}\} \supset \{\text{algebraic action}\}.\end{aligned} \tag{42}$$

An important structure in any of these contexts is *reducibility*. Suppose that we are in a linear situation. A representation (or a linear action) of a group G on a vector space V is called *completely reducible* if it is a direct sum of irreducible representations.

If G is a compact topological group acting linearly on a (real or complex) vector space V then the action is completely reducible. Thus, compactness is enough to guarantee complete irreducibility. But, sometimes we do not want to assume compactness.

Definition 3.2.1. A complex Lie group G is said to be ***reductive*** if

- G has finitely many connected components.

- G contains a compact real Lie subgroup K such that $\mathbf{C} \otimes_{\mathbf{R}} T_e K \simeq T_e G$.

The compact group K is called the reductive form of G.

Example 3.2.2. A basic example of a reductive group is the group $GL(n, \mathbf{C})$ of invertible complex-valued $n \times n$ matrices. In this case the real subgroup $U(n)$ can be taken to be the group of unitary $n \times n$ matrices.

THEOREM 3.2.3. *A reductive linear group action on a complex vector space is completely reducible.*

Let's turn to the nice algebraic context that we will work in for the time being.

Definition 3.2.4. A ***linear algebra group*** G is a subgroup of $GL(V)$, where V is some complex vector space, which is cut out by a finite collection of polynomials. That is, there exists $p_1, \ldots, p_k \in \mathbf{C}[\mathrm{End}(V)]$ such that

$$G = \{g \in GL(V) \mid p_i(g) = 0, \text{for all } i\}. \tag{43}$$

Example 3.2.5. The groups $GL(n, \mathbf{C})$, $SL(n, \mathbf{C})$, $O(n, \mathbf{C})$, $SO(n, \mathbf{C})$, $Sp(2n, \mathbf{C})$ are all linear algebra groups. The group $U(n)$ is not a complex linear algebraic group, though it is a real algebraic group.

Definition 3.2.6. An ***affine algebraic group*** G is an affine algebraic variety together with structure maps $\mu \colon G \times G \to G$, $(-)^{-1} \colon G \to G$, and an element $e \in G$ such that the usual group axioms hold.

An ***algebraic action*** of an algebraic group on an affine algebraic variety X is a map of algebraic varieties $G \times X \to X$ satisfying the axioms of an action.

Proposition 3.2.7. *Any affine algebraic group is isomorphic (as an affine algebraic group) to a linear algebraic group.*

We say an algebraic group G is reductive if it is reductive as a complex Lie group (in the sense above).

Example 3.2.8. The groups $GL(n, \mathbf{C})$, $SL(n, \mathbf{C})$, $O(n, \mathbf{C})$, $SO(n, \mathbf{C})$, $Sp(2n, \mathbf{C})$ are reductive. An example of a non-reductive group is the additive group $\mathbb{G}_a = (\mathbb{C}, +)$. Another example of a non-reductive group is the Borel subgroup of $GL(n, \mathbf{C})$ consisting of upper triangular matrices with 1's along the diagonal.

Here is the main theorem about reductive group actions.

THEOREM 3.2.9. *Let G be a reductive algebraic group acting on an affine variety X. Then*

- $\mathbf{C}[X]^G$ *is a finitely generated* $\mathbf{C}$*-algebra.*
- *If $W, Z \subset X$ are closed, G-invariant, and disjoint then there exists a G-invariant polynomial function $p \in \mathbf{C}[X]$ such that $p|_W \equiv 0$ and $p|_Z \equiv 1$.*

3.3. QUOTIENTS IN AFFINE ALGEBRAIC GEOMETRY

For the next few lectures we will be working in the context where G is an algebraic group acting on an affine algebraic variety X (all defined over $\mathbf{C}$). We will consider X as a topological space using the Zariski topology.

Some key results of invariant theory which we will not prove include the following.

- Every G-orbit in X is a nonsingular algebraic variety (which is not necessarily closed).
- For every orbit $\mathbb{O}$, we can consider the closure $\overline{\mathbb{O}}$. The boundary $\overline{\mathbb{O}} - \mathbb{O}$ is a union of lower dimensional orbits.

In the last lecture we introduced the so-called *geometric invariant theory*, or simply *GIT*, quotient

$$X /\!/ G \stackrel{\text{def}}{=} \operatorname{Spec}\left(\mathbf{C}[X]^G\right). \tag{44}$$

By the first part of theorem 3.2.9 this is an affine algebraic variety. Also, there is a relationship between the set-theoretic quotient M/G and the GIT quotient. Indeed, given any orbit $\mathbb{O} \in X/G$ we can define the maximal ideal

$$J_{\mathbb{O}} \stackrel{\text{def}}{=} \{f \in \mathbf{C}[X]^G \mid f|_{\mathbb{O}} = 0\}. \tag{45}$$

The assignment $\mathbb{O} \mapsto J_{\mathbb{O}}$ defines a continuous map

$$p\colon X/G \to X /\!/ G, \tag{46}$$

where $X /\!/ G$ is equipped with the Zariski topology.

Aside 3.3.1. The Zariski topology is the natural topology on schemes. Let's consider the affine world. The closed sets in $\mathbf{A}^n$ are precisely the algebraic sets. That is, those sets of the form

$$\{x \in \mathbf{A}^n \quad f(x) = 0, f \in S\} \tag{47}$$

where S is some set of polynomials.

The Zariski topology, while natural from the point of view of algebraic geometry, is rather pathological. For example, it is rarely Hausdorff. Consider $\mathbf{A}^1 = \operatorname{Spec}(\mathbf{C}[x])$. Since all prime ideals are maximal, every point in $\mathbf{A}^1$ is closed in the Zariski topology. Nevertheless, it fails to be Hausdorff.

THEOREM 3.3.2. *For X affine and reductive this map is surjective. Moreover two orbits $\mathbb{O}, \mathbb{O}'$ determine the same point in $X /\!/ G$ if and only if the closures of the orbits are disjoint $\overline{\mathbb{O}} \cap \overline{\mathbb{O}}' \neq \emptyset$.*

Thus, as a topological space $X /\!/ G$ is $X/\sim$ where $x \sim x'$ if and only if $\mathbb{O}_x \cap \mathbb{O}_{x'} \neq \emptyset$.

From this we obtain an explicit description of the GIT quotient.

THEOREM 3.3.3. *There is a homeomorphism of topological spaces*

$$X /\!/ G \simeq \{\text{closed orbits in } M\} \tag{48}$$

which sends $[x] \mapsto$ the unique closed orbit contained in $\overline{\mathbb{O}}_x$.

PROOF. It suffices to show that the closure of any orbit contains a unique closed orbit. For the existence of a closed orbit, recall that boundary of an orbit $\overline{\mathbb{O}} - \mathbb{O}$ is a union of orbits of lower dimension. For uniqueness one relies on Theorem 3.2.9. ☐

As a corollary of this we see that if all orbits are closed then the GIT quotient agrees with the set-theoretic one. Here is a simple example where there is a non-closed orbit.

Example 3.3.4. Suppose that $X = \mathbf{A}^2$ and consider the scaling action of $G = \mathbf{C}^\times$

$$\lambda \cdot (z_1, z_2) = (\lambda z_1, \lambda z_2). \tag{49}$$

Then as a topological space one has

$$\mathbf{A}^2/\mathbf{C}^\times \simeq \mathbb{P}^1 \cup \{0\}. \tag{50}$$

(Note that $\mathbf{A}^2/\mathbf{C}^\times$ is not a manifold.) On the other hand, the only closed orbit is $\{0\} \subset \mathbf{A}^2$. Indeed, if $0 \neq x \in \mathbf{A}^2$ then $\mathbb{O}_x = \ell_x - \{0\}$ where ℓ_x is the line through x and 0. Thus

$$\mathbf{A}^2 /\!/ \mathbf{C}^\times \simeq \{0\}. \tag{51}$$

Notice that this is consistent with the fact $\mathbf{C}[z_1, z_2]^{\mathbf{C}^\times} = \mathbf{C}$.

In the topological world, we obtain the best structure on the quotient when the G-action is free. It is in this case that the quotient is a smooth manifold (if we start with a smooth manifold) and the quotient map exhibits the original space as a principal bundle. There is an analog of this result in the algebro-geometric world.

To formulate we need to have an algebro-geometric version of a smooth principal bundle. This is a bit outside of the scope of the topics of the course. If $x \in X$ is a point in an algebraic variety then let $\mathcal{O}_x$ be the completed local ring at $x \in X$. By definition, this is the stalk of the sheaf $\mathcal{O}_X$ at x

$$\mathcal{O}_{X,x} = \lim_{U \ni x} \mathcal{O}_X(U). \tag{52}$$

In other words, this is the ring of ***germs of functions*** at $x \in X$. This is a local ring with maximal ideal given by functions which vanish at p. We denote by $\widehat{\mathcal{O}_{X,x}}$ the completion of this local ring.

Example 3.3.5. Suppose that $X = \mathrm{Spec}\,(\mathbf{C}[z_1, \ldots, z_n]/(f))$ where f is some polynomial. Then $\widehat{\mathcal{O}_0}$ is isomorphic to $\mathbf{C}[[z_1, \ldots, z_n]]/(f)$.

For us, the analog of a principal G-bundle in algebraic geometry will be an étale G-bundle. While we won't give the careful definition of étale, let's point out a useful feature.

A G-equivariant map $f \colon X \to Y$ is an étale G-bundle if for every $y \in Y$ there exists an open neighborhood $U \subset Y$ of y such that $X|_U \to U$ is étale equivalent to the trivial G-bundle $U \times G \to U$.

A morphism of algebraic varieties $f \colon X \to Y$ is called *étale* if for every $x \in X$ the pullback map

$$f^* \colon \widehat{\mathcal{O}_{f(x)}} \to \widehat{\mathcal{O}_x} \tag{53}$$

is an isomorphism.

THEOREM 3.3.6 (Luna Slice). *Suppose that X is a non-singular affine variety equipped with a free G-action where G is a reductive algebraic group. Then $X /\!/ G = X/G$ is a non-singular variety and $X \to X /\!/ G$ is an étale G-bundle.*

3.4. A FUNDAMENTAL EXAMPLE

We end this lecture with a very important example. Let

$$X = \mathrm{End}(\mathbf{C}^n) = \mathbf{A}^{n^2} \tag{54}$$

be the affine space of $n \times n$ matrices. Consider the adjoint action of $G = GL(n, \mathbf{C})$ on X.

Given a matrix $x \in X$ consider its characteristic polynomial

$$\det(x + t\mathbb{1}) = p_n(x) + tp_{n-1}(x) + \cdots + t^{n-1}p_1(x) + t^n \tag{55}$$

where t is some indeterminate.

THEOREM 3.4.1. *The polynomials $p_1, \ldots, p_n \in \mathbf{C}[X]$ are algebraically independent, $GL(n, \mathbf{C})$-invariant, and generate $\mathbf{C}[\mathrm{End}(\mathbf{C}^n)]^{GL(n,\mathbf{C})}$. In particular*

$$\mathrm{End}(\mathbf{C}^n) \,/\!/\, GL(n, \mathbf{C}) \simeq \mathbf{A}^n. \tag{56}$$

The proof of this theorem relies on the following fundamental lemma.

Lemma 3.4.2. *Suppose that an algebraic group G acts on a variety X and suppose that $p_1, \ldots, p_n \in \mathbf{C}[X]$ are G-invariant polynomials. Further, suppose that $H \subset G$ is a subgroup which leaves invariant a subvariety $U \subset X$ with the properties:*

(1) *the polynomials $p_i|_U$ generate the ring $\mathbf{C}[U]^H$;*
(2) *the set $G \cdot U$ is dense in X.*

Then the polynomials $p_1, \ldots, p_n$ generate the ring $\mathbf{C}[X]^G$.

PROOF. Let p be a polynomial which is G-invariant. Then, by assumption,

$$p|_U = F(p_1|_U, \ldots, p_n|_U) \tag{57}$$

for some polynomial F. Define

$$q \stackrel{\text{def}}{=} p - F(p_1, \ldots, p_n) \in \mathbf{C}[X]. \tag{58}$$

By construction $q|_U \equiv 0$. Moreover, q is G-invariant so that $q|_{G \cdot U} \equiv 0$. By the second assumption we know that $q \equiv 0$. □

Let's turn to the proof of theorem 3.4.1

PROOF OF THEOREM 3.4.1. Fix a basis $\{e_i\}$ of $\mathbf{C}^n$. We apply the lemma to the case where $U \subset X = \mathrm{End}(\mathbf{C}^n)$ is the subspace of diagonal matrices and $H = S_n$ is the permutation group which permutes the $e_1, \ldots, e_n$. In particular, a permutation $\sigma \in S_n$ acts on U by the rule

$$\sigma \cdot \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} = \begin{pmatrix} \lambda_{\sigma(1)} & & \\ & \ddots & \\ & & \lambda_{\sigma(n)} \end{pmatrix} \tag{59}$$

We will first show that $\overline{G \cdot U} = X$. Consider the continuous map $\pi \colon X \to \mathbf{C}^n$ which sends a matrix x to the coefficients $p_1(x), \ldots, p_n(x)$ of its characteristic polynomial. This map can be shown to be surjective. The subset $V \subset \mathbf{C}^n$ consisting of the coefficients of monic polynomials with distinct roots is an open set in the Euclidean topology on affine space. Thus, $\pi^{-1}(V) \subset X$ is an open set which consists of matrices with distinct eigenvalues.[1] Finally, we note that any matrix which has distinct eigenvalues can be diagonalized.

Next, we must show that $p_1|_U, \ldots, p_n|_U$ generate $\mathbf{C}[U]^{S_n}$. Let $x \in X$ be the matrix $\mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. Then the characteristic polynomial of x is

$$\det(x + t\mathbb{1}) = \sigma_n(\lambda) + \cdots + \sigma_1(\lambda)t^{n-1} + t^n \tag{60}$$

where σ_i is the ith symmetric polynomial. We have already recalled in the last lecture that the elementary symmetric polynomials generate the ring of all symmetric functions. This completes the proof. □

[1] An open set in the Euclidean topology on affine space is automatically dense in the Zariski topology.

LECTURE 4

An explicit description of the Hilbert scheme

Today we will prove an explicit characterization of the Hilbert scheme. Let

$$H_n \subset \mathrm{Hom}(\mathbf{C}^n, \mathbf{C}^n)^{\oplus 2} \oplus \mathbf{C}^n \oplus (\mathbf{C}^n)^* \tag{61}$$

is the set of tuples (X, Y, i, j) which satisfy

$$[X, Y] + ij = 0. \tag{62}$$

Also define

$$H_n^s \subset H_n \tag{63}$$

to be the subspace where

- the vector $i(1)$ generates $\mathbf{C}^n$ under the action by X, Y. This means that given $v \in \mathbf{C}^n$ there exists integers $k, l \geq 0$ such that $v = X^k Y^l i(1)$.

This last condition is called a ***stability*** condition. We will see many versions of it in future lectures.

THEOREM 4.0.1. *For any n the Hilbert scheme $\mathrm{Hilb}_n(\mathbf{A}^2)$ is a nonsingular algebraic variety of dimension $2n$. Moreover, there is an isomorphism of algebraic varieties*

$$\mathrm{Hilb}_n(\mathbf{A}^2) \simeq H_n^s \,/\!/\, GL(n, \mathbf{C}). \tag{64}$$

There is a similar description of the symmetric product of $\mathbf{A}^2$.

THEOREM 4.0.2. *There is an isomorphism of algebraic varieties*

$$S^n \mathbf{A}^2 \simeq H_n \,/\!/\, GL(n, \mathbf{C}). \tag{65}$$

Moreover, the natural map $H_n^s \hookrightarrow H_n$ induces the Hilbert–Chow morphism

$$\pi_{HC} \colon \mathrm{Hilb}_n(\mathbf{A}^2) \to S^n \mathbf{A}^2. \tag{66}$$

4.1. A description of the symmetric product

We will begin with the description of the symmetric product which means we will momentarily forget about the stability condition. First, let's carefully describe how $GL(n, \mathbf{C})$ acts on H_n. The action is a restriction of the most natural one where $GL(n; \mathbf{C})$ acts on endomorphisms of $\mathbf{C}^n$ by conjugation and acts on $\mathbf{C}^n$ (respectively $(\mathbf{C}^n)^*$) in the defining (respectively antidefining) way. Explicitly, for g an invertible $n \times n$ matrix and $(X, Y, i) \in H_n'$ the action is

$$g \cdot (X, Y, i, j) \stackrel{\text{def}}{=} \left(gXg^{-1}, gYg^{-1}, gi, jg^{-1}\right). \tag{67}$$

The following lemma is a direct calculation and can be found in [Nak99, §2].

Lemma 4.1.1. *Suppose $[X,Y]+ij=0$ as above. Let $S\subset \mathbf{C}^n$ be the subset*

$$\sum (X^{n_1}Y^{m_1}\cdots X^{n_k}Y^{m_k})i(\mathbf{C}). \tag{68}$$

Then $j|_S\equiv 0$.

Suppose that

$$\mathbb{O} \overset{\text{def}}{=} GL(n,\mathbf{C})\cdot (X,Y,i,j) \tag{69}$$

is a closed orbit. Using $S\subset \mathbf{C}^n$ as in the lemma, we can decompose

$$\mathbf{C}^n = S\oplus S^{\perp}. \tag{70}$$

With respect to this decomposition the matrices (X,Y,i,j) have the form

$$X,Y=\begin{pmatrix}\star & \star\\ 0 & \star\end{pmatrix},\quad i=\begin{pmatrix}\star\\ 0\end{pmatrix},\quad j=\begin{pmatrix}0 & \star\end{pmatrix}. \tag{71}$$

By closedness we can further assume that

$$X,Y=\begin{pmatrix}\star & 0\\ 0 & \star\end{pmatrix},\quad i=\begin{pmatrix}0\\ 0\end{pmatrix},\quad j=\begin{pmatrix}0 & 0\end{pmatrix}. \tag{72}$$

Thus, the condition $[X,Y]+ij=0$ simply becomes $[X,Y]=0$. Choose a basis so that X,Y are both upper triangular. Then, by the closedness assumption we can assume that X,Y are diagonalizable. The equivalence with $S^n\mathbf{A}^2$ associates a closed orbit to the simultaneous eigenvalues of the matrices X,Y.

Remark 4.1.2. The above argument can be modified to give a short proof that $\mathrm{End}(\mathbf{C}^n) /\!/ GL(n,\mathbf{C})\simeq \mathbf{A}^n$. Indeed, every matrix admits a basis for which it is upper triangular. If we assume a matrix lies in a closed orbit, then we can also assume it is diagonalizable. The equivalence then sends a closed $GL(n,\mathbf{C})$-orbit to its n-tuple of eigenvalues.

4.2. Relating closed points

Before turning to the proof of theorem 4.0.1, we will give a heuristic argument for the result. First, it turns out that we can simplify the description of H_n.

Lemma 4.2.1. *Suppose that $(X,Y,i,j)\in H_n^s$. Then $j=0$.*

Let $\widetilde{H}_n^s$ be the subspace consisting of (X,Y,i) with the property that $[X,Y]=0$ and i generates $\mathbf{C}^n$ under the action by X,Y. Then as a corollary of this lemma we have $GL(n;\mathbf{C})$-equivariant isomorphism

$$\widetilde{H}_n^s\simeq H_n^s. \tag{73}$$

Let's see how the data of a triple $(X,Y,i)\in \widetilde{H}_n^s$ gives rise to a closed point in $\mathrm{Hilb}_n(\mathbf{A}^2)$. Notice that a closed point in $\mathrm{Hilb}_n(\mathbf{A}^2)$ is, by definition, an ideal I in $\mathbf{C}[z_1,z_2]$ such that $\mathbf{C}[z_1,z_2]/I$ is an n-dimensional vector space. Define the linear map

$$\phi_{(X,Y,i)}\colon \mathbf{C}[z_1,z_2]\to \mathbf{C}^n \tag{74}$$

by the formula $\phi_{(X,Y,i)}(f) = f(X,Y)i(1)$. Since $\operatorname{im}\phi$ is invariant under the action of X, Y and contains $\operatorname{im} i$ we see that ϕ is surjective by the stability condition. Thus $I = \ker\phi$ is an ideal in $\mathbf{C}[z_1, z_2]$ and $\dim_{\mathbf{C}}(\mathbf{C}[z_1, z_2]/I) = n$.

Next, suppose I is an ideal of codimension n and let $V = \mathbf{C}[z_1, z_2]/I$. Then we have operators $X = z_1$, $Y = z_2$ and $i\colon \mathbf{C} \to V$ defined by $i(1) = 1 \mod I$. It is automatic that $[X, Y] = 0$ and that the stability condition holds.

It is not hard to check that these two operations are mutually inverse to one another, which thus gives an isomorphism of *sets* between $\widetilde{H}_n^s/GL(n, \mathbf{C})$ and codimension n ideals in $\mathbf{C}[z_1, z_2]$.

4.3. Proof of the theorem

By the isomorphism (73), we see that theorem 4.0.1 follows from the following result.

Proposition 4.3.1. *The Hilbert scheme of n-points on affine space* $\mathrm{Hilb}_n(\mathbf{A}^2)$ *is isomorphic to the nonsingular algebraic variety*

$$\widetilde{H}_n^s \,/\!/\, GL(n, \mathbf{C}). \tag{75}$$

PROOF. We will use the algebraic slice theorem as formulated in the previous lecture to argue why $\widetilde{H}_n^s \,/\!/\, GL(n; \mathbf{C})$ is nonsingular. Before taking the quotient, we need to see that H_n^s is non-singular. Consider the map

$$F\colon \mathrm{End}(\mathbf{C}^n)^{\otimes 2} \otimes \mathbf{C}^n \to \mathrm{End}(\mathbf{C}^2) \tag{76}$$

defined by $F(X, Y, i) = [X, Y]$. Let S be the subset of $\mathrm{End}(\mathbf{C}^n)^{\otimes 2} \otimes \mathbf{C}^n$ consisting of triples (X, Y, i) satisfying the stability condition. Observe that $\widetilde{H}_n^s = (F|_S)^{-1}(0)$. To show that $\widetilde{H}_n^s$ is non-singular it suffices to show that the derivative of $F|_S$ has constant rank.

Lemma 4.3.2. *Let* $D = D_{(X,Y,i)}(F|_S)$ *be the derivative of the map* $F|_S$ *at* $(X, Y, i) \in S$. *Then*

$$\operatorname{coker} D = \{A \in \mathrm{End}(\mathbf{C}^n) \mid [X, A] = [Y, A] = 0\}. \tag{77}$$

Using this description we can define a map $\operatorname{coker} D \to \mathbf{C}^n$ by the rule $A \mapsto A(i(1))$. Conversely, define a map $\mathbf{C}^n \to \operatorname{coker} D$ by sending v to the endomorphism A which satisfies

$$A_v(X^k Y^l i(1)) = X^k Y^l v \tag{78}$$

for integers $k, l \geq 0$. This is enough to define A_v by the stability condition. These maps are clearly mutual inverses so that $\operatorname{coker} D \simeq \mathbf{C}^n$. Thus, by the constant rank level set theorem (see [Lee13][Theorem 5.12]) we see that $\widetilde{H}_n^s$ is non-singular.

Next, we will apply Luna's slice theorem to see that $\widetilde{H}_n^s \,/\!/\, GL(n, \mathbf{C})$ is non-singular. For this we need to check that the action is free. Suppose that $g \in GL(n, \mathbf{C})$ stabilizes (X, Y, i). This means that

$$gXg^{-1} = X, \quad gYg^{-1} = Y, \quad gi = i. \tag{79}$$

The last equality implies that $\ker(g - \mathbb{1}) \subset \mathbf{C}^n$ contains $\operatorname{im} i$. But the first two equations imply that this subspace is stabilized by X, Y. Thus by the stability condition $g = \mathbb{1}$. By the slice theorem

$$Y \overset{\text{def}}{=} \widetilde{H}_n^s /\!/ GL(n, \mathbf{C}) = \widetilde{H}_n^s / GL(n, \mathbf{C}) \tag{80}$$

is a non-singular variety and the map

$$\widetilde{H}_n^s \to Y \tag{81}$$

is an étale principal $GL(n, \mathbf{C})$-bundle.

Now, to characterize Y as the Hilbert scheme of n-points on $\mathbf{A}^2$ we need to construct a universal family over Y of 0-dimensional subschemes of size n. There is a family on $\mathcal{Y} \to Y$ defined by the natural surjection

$$f(z_1, z_2) \in \mathbf{C}[z_1, z_2] \mapsto f(X, Y)i(1) \in \mathbf{C}^n. \tag{82}$$

To see that it is a universal family we need to show that if $\pi\colon Z \to U$ is any flat family of 0-dimensional closed subschemes of $\mathbf{A}^2$ of size n, then there exists a unique morphism $\phi\colon U \to Y$ fitting into the pullback square

$$\begin{array}{ccc} Z & \longrightarrow & \mathcal{Y} \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ U & \xrightarrow[\phi]{} & Y. \end{array} \tag{83}$$

By assumption $\pi_* \mathcal{O}_Z$ is a locally free sheaf of rank n on U. Just as in the previous section, define X, Y as the $\mathcal{O}_U$-linear operators acting on $\pi_*\mathcal{O}_Z$ given by multiplication by the coordinate functions z_1, z_2 respectively. Also let i be the image of the constant polynomial 1 thought of as an sheaf homomorphism $\mathcal{O}_U \to \pi_*\mathcal{O}_Z$. If we fix an open cover $U = \cup_\alpha U_\alpha$ so that $\pi_*\mathcal{O}_Z$ is trivializable over U_α then we obtain morphisms $U_\alpha \to \widetilde{H}_n^s$ for each α. Composing with $\widetilde{H}_n^s \to Y$ these glue together to define a morphism $\phi\colon U \to Y$. By construction we have $\phi^*\widetilde{Y} = Z$. □

Remark 4.3.3. Notice that the proof of this result didn't rely much of our knowledge of the GIT quotient. Since the $GL(n, \mathbf{C})$ action on $\widetilde{H}_n^s$ is free, the GIT quotient is the same as the set-theoretic quotient.

LECTURE 5

Moduli spaces of sheaves, I

Last time we showed that the Hilbert scheme of n points in $\mathbf{A}^2$ is non-singular and equivalent to the quotient of

(84) $$\widetilde{H}_n^s = \{(X, Y, i) \mid [X, Y] = 0, \quad \text{and stability}\} \subset \operatorname{End}(\mathbf{C}^n)^{\oplus 2} \oplus \mathbf{C}^n.$$

by the natural $GL(n, \mathbf{C})$ action. Today we will wrap up this discussion with a computation of the dimension of $\operatorname{Hilb}_n(\mathbf{A}^2)$ and some examples of Hilbert schemes for small values of n. Then, we turn to a sheaf-theoretic description of the Hilbert scheme.

5.1. Dimension of the Hilbert scheme

For $(X, Y, i) \in \widetilde{H}_n^s$ let $(C^\bullet, \mathrm{d})$ be the following complex

(85) $$\operatorname{End}(\mathbf{C}^n) \xrightarrow{\mathrm{d}_1} \operatorname{End}(\mathbf{C}^n)^{\oplus 2} \oplus \mathbf{C}^n \xrightarrow{\mathrm{d}_2} \operatorname{End}(\mathbf{C}^n)$$

where the first arrow is the derivative of the $GL(n, \mathbf{C})$ action

(86) $$\mathrm{d}_1(A) = ([A, X], [A, Y], Ai)$$

and the second arrow is

(87) $$\mathrm{d}_2(A, B, v) = [X, A] + [Y, B].$$

Then the tangent space at (X, Y, i) is

(88) $$\mathrm{T}_{(X,Y,i)} \operatorname{Hilb}_n(\mathbf{A}^2) \simeq H^1(C, \mathrm{d}).$$

We have already shown that the dimension of the cokernel of d_2 is n. By the stability condition we have $\ker \mathrm{d}_1 = 0$. This shows that $\dim H^1(C) = 2n$.

5.2. Examples

Let's consider some examples of $\operatorname{Hilb}_n(\mathbf{A}^2)$ for small n. For $n = 1$ we have $X = x, Y = y$ for some numbers $x, y \in \mathbf{C}$. Furthermore, the stability condition implies that $i \neq 0$. Using the $\mathbf{C}^\times$-action we can assume that $i = 1$. The ideal corresponding to the pair x, y is

(89) $$I = \{f(z_1, z_2) \in \mathbf{C}[z_1, z_2] \mid f(x, y) = 0\}.$$

This is simply the maximal idea corresponding to $(x, y) \in \mathbf{A}^2$. Thus $\operatorname{Hilb}_1(\mathbf{A}^2) = \mathbf{A}^2$.

Next we look at $n = 2$. Then X, Y are 2×2 matrices. Suppose that at least one of X, Y have distinct eigenvalues. Since $[X, Y] = 0$ we can assume that

$$X = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix} \tag{90}$$

with $(x_1, y_1) \neq (x_2, y_2)$. By the stability condition we can take

$$i(1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{91}$$

The corresponding ideal is

$$I = \{f(z_1, z_2) \in \mathbf{C}[z_1, z_2] \mid f(x_i, y_i) = 0\}, \tag{92}$$

which corresponds to two distinct points in $\mathbf{A}^2$. Thus away from the diagonal in $\mathbf{A}^2 \times \mathbf{A}^2$ the Hilbert scheme agrees with $S^2\mathbf{A}^2$.

The interesting stuff happens when we assume that X, Y each have one eigenvalue. We cannot assume that X, Y are both diagonalizable as this violates the stability condition. Thus, we have

$$X = \begin{pmatrix} x & a \\ 0 & x \end{pmatrix}, \quad Y = \begin{pmatrix} y & b \\ 0 & y \end{pmatrix} \tag{93}$$

for some $(a, b) \in \mathbf{A}^2 - 0$. In this basis we can assume that

$$i(1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{94}$$

The corresponding ideal is

$$I = \left\{ f(z_1, z_2) \in \mathbf{C}[z_1, z_2] \mid f(x, y) = \left(a\frac{\partial f}{\partial z_1} + b\frac{\partial f}{\partial z_2} \right)(x, y) = 0 \right\}. \tag{95}$$

This corresponds to two infinitesimally close points in $\mathbf{A}^2$ at (x, y) which point to each other in the direction of the vector field $a\frac{\partial}{\partial z_1} + b\frac{\partial}{\partial z_2}$.

5.3. Torsion-free sheaves

A quasi-coherent sheaf $\mathcal{F}$ on an algebraic variety X is ***torsion-free*** if for every affine open subset $U \subset X$ the space of local sections $\mathcal{F}(U)$ is torsion-free as a module over the ring of functions $\mathcal{O}(U)$ on U. That is, for ever nonzero section $s \in \mathcal{F}(U)$ and nonzero function $f\colon U \to \mathbf{C}$ one has $f \cdot s \neq 0$. A typical example of a torsion-free sheaf is the sheaf of sections of a vector bundle; the condition of being a locally free implies torsion-free. We will mostly be concerned with coherent torsion-free sheaves.

For any quasi-coherent sheaf $\mathcal{F}$ there is a canonical morphism

$$\mathcal{F} \to (\mathcal{F}^\vee)^\vee = \mathcal{F}^{\vee\vee} \tag{96}$$

where $\mathcal{F}^\vee = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ is the dual sheaf.[1] The main technical result about torsion-free sheaves that we will use is the following.

THEOREM 5.3.1 (**[??]**). *Let X be a non-singular algebraic variety and suppose $\mathcal{F}$ is a coherent torsion-free sheaf on X. Then:*

[1]A quasi-coherent sheaf which is isomorphic to its double dual is called reflexive.

- *there exists a Zariski open set $U \subset X$ of codimension ≥ 2 such that $\mathcal{F}|_U$ is locally free.*
- *If $\dim X = 2$ then the sheaf $\mathcal{F}^{\vee\vee}$ is locally free of finite rank and the morphism $\mathcal{F} \to \mathcal{F}^{\vee\vee}$ is injective. Restriction of this morphism to U results in an isomorphism $\mathcal{F}|_U \xrightarrow{\sim} \mathcal{F}^{\vee\vee}|_U$.*

By the second item we see that any torsion-free sheaf is a subsheaf of a coherent locally free sheaf. Consider the dual $\mathcal{F}^\vee$ and write this as the quotient of a locally free sheaf $\mathcal{E}$. Then $\mathcal{F}^{\vee\vee}$ is a subsheaf of $\mathcal{E}^\vee$ which is still locally free.

Example 5.3.2. Suppose that $\mathcal{F}$ is a rank one torsion-free sheaf on a surface S. Since $\mathcal{F}^{\vee\vee}$ is locally free it is a line bundle.

Example 5.3.3. Suppose that $J \in \mathrm{Hilb}_n(X)$ where X is any affine variety (not necessarily of dimension two). Let $\mathcal{F}_J$ be the corresponding ideal sheaf of $\mathcal{O}_X$ which satisfies $\Gamma(X, \mathcal{F}_J) = J$. Then $\mathcal{F}_J$ is torsion-free and $\mathcal{O}/\mathcal{F}_J$ is a torsion sheaf

$$\Gamma(X, \mathcal{O}_X/J) = \mathbf{C}[X]/J. \tag{97}$$

Example 5.3.4. Consider the ideal sheaf of the point $0 \in \mathbf{A}^2$. This sheaf is torsion-free but not locally free.

Example 5.3.5. Consider the morphism of sheaves on $\mathbf{A}^2$

$$\phi \colon \mathcal{O} \to \mathcal{O}^{\oplus 2} \tag{98}$$

defined by $\Phi(f) = (z_1 f, z_2 f)$. Then ϕ is injective and its image is

$$\mathrm{im}(\phi) = \{(f_1, f_2) \mid z_1 f_2 = z_2 f_1\} \subset \mathcal{O}^{\oplus 2}. \tag{99}$$

We claim that

$$\mathcal{F} \stackrel{\mathrm{def}}{=} \mathrm{coker}\, \phi \tag{100}$$

is a torsion-free sheaf. Indeed let $\psi \colon \mathcal{O}^{\oplus 2} \to \mathcal{O}$ be $\psi(g_1, g_2) = z_2 g_1 - z_1 g_2$. Then $\ker \psi = \mathrm{im}\, \phi$ so

$$\mathcal{F} \simeq \mathrm{im}\, \psi = \{f \in \mathcal{O} \mid f(0,0) = 0\}. \tag{101}$$

By this last equivalence we see that $\mathcal{F}$ is isomorphic to a subsheaf of $\mathcal{O}$.

More generally one has the following. Let V be a finite dimensional vector space and $A_1, A_2 \in \mathrm{End}(V)$. Denote by $\mathcal{V} = \mathcal{O} \otimes V$ the trivial vector bundle on $\mathbf{A}^2$ with fiber V. Define

$$\phi \colon \mathcal{V} \to \mathcal{V} \otimes \mathbf{C}^2 \tag{102}$$

by $s \mapsto ((A_1 - z_1)s, (A_2 - z_2)s)$.

Lemma 5.3.6. *The sheaf* $\mathrm{coker}\, \phi$ *is torsion-free.*

LECTURE 6

Moduli spaces of sheaves, II

We are heading towards a definition of the Hilbert scheme in terms sheaves.

6.1. Chern classes

Let X be a smooth algebraic variety over $\mathbf{C}$, which you are free to think of as just complex manifold. The jth Chern class of a complex vector bundle E over X is an element

(103) $$c_j(E) \in H^{2j}(X; \mathbf{R}).$$

The total Chern class is usually denoted

(104) $$c(E) = \sum_{j \geq 0} c_j(E) \in H^{2\bullet}(X; \mathbf{R}),$$

or its one parameter version

(105) $$c_t(E) = \sum_{j \geq 0} t^j c_j(E) \in H^{2\bullet}(X; \mathbf{R})[t].$$

The Chern classes are determined by the following axioms.

- *The zeroeth Chern class*. For any bundle $E \to X$ one has $c_0(E) = 1$.
- *Naturality*. For any bundle $E \to X$ and smooth map $f\colon Y \to X$ one has

(106) $$c(f^*E) = f^*c(E) \in H^{2\bullet}(X; \mathbf{R}).$$

- *Whitney sum.* For a finite collection of bundles E_i one has

(107) $$c(\oplus_i E_i) = \prod_i c(E_i).$$

- *Normalization.* Let $\mathcal{O}(1)$ be the dual of the tautological line bundle over $\mathbf{CP}^1$. Then

(108) $$\int_{\mathbf{CP}^1} c_1(\mathcal{O}(1)) = 1.$$

We will need to extend the definition of Chern classes to coherent sheaves. Let $\mathrm{Coh}(X)$ be the category of coherent sheaves on X and let $\mathrm{Vect}(X) \subset \mathrm{Coh}(X)$ be the subcategory of locally free coherent sheaves. This subcategory is equivalent to the category of holomorphic vector bundles on X; the equivalence is obtained by taking the sheaf of holomorphic sections of a given holomorphic vector bundle. Both $\mathrm{Coh}(X)$ and $\mathrm{Vect}(X)$ are abelian categories.

Construction 6.1.1. Given any abelian category $\mathcal{A}$ we can look at the free abelian group $\mathbf{Z}[\mathcal{A}]$ which is generated by the isomorphism classes of objects of $\mathcal{A}$. Given a short exact sequence

$$0 \to A \to B \to C \to 0 \tag{109}$$

in $\mathcal{A}$ we can form the element

$$-[A] + [B] - [C] \in \mathbf{Z}[\mathcal{A}]. \tag{110}$$

Let $E(\mathcal{A})$ be the subgroup of $\mathbf{Z}[\mathcal{A}]$ generated by elements of this form. The ***Grothendieck group*** of the abelian category $\mathcal{A}$ is defined as the quotient group

$$K_0(\mathcal{A}) \stackrel{\text{def}}{=} \mathbf{Z}[\mathcal{A}]/E(\mathcal{A}). \tag{111}$$

By definition, if (109) is a short exact sequence then we have the relation

$$[B] = [A] + [C] \tag{112}$$

in $K_0(\mathcal{A})$.

If $\mathcal{A}_0 \subset \mathcal{A}$ is an additive (not necessarily full) subcategory which is closed under extensions then the above definition endows $K_0(\mathcal{A}_0)$ also with the structure of an abelian group. Such an $\mathcal{A}_0$ is called an ***exact*** category.

We apply this construction to the situation

$$\mathcal{A}_0 = \text{Vect}(X) \subset \text{Coh}(X) = \mathcal{A}. \tag{113}$$

Notice that tensor product endows both $K_0(X) = K_0(\text{Vect}(X))$ with the structure of a commutative ring.

Lemma 6.1.2. *Let X be a smooth complex variety or a complex manifold.*

(1) The subring $E(\text{Vect}(X)) \subset \mathbf{Z}[\text{Vect}(X)]$ is an ideal, and therefore $K_0(X)$ has the structure of a commutative ring with unit given by the trivial rank one vector bundle.
(2) The group $K_0(\text{Coh}(X))$ is naturally a module for $K_0(X)$.
(3) The embedding $\mathbf{Z}[\text{Vect}(X)] \hookrightarrow \mathbf{Z}[\text{Coh}(X)]$ determines a group homomorphism

$$i\colon K_0(X) \to K_0(\text{Coh}(X)). \tag{114}$$

By the axioms of Chern classes above, we see that the total Chern class defines a group homomorphism

$$c\colon K_0(X) \to H^\bullet(X). \tag{115}$$

An immediate consequence of this is a slightly more general version of the Whitney sum axiom. If we have any exact sequence of vector bundles

$$0 \to E' \to E \to E'' \to 0 \tag{116}$$

then

$$c_t(E) = c_t(E') \cdot c_t(E''). \tag{117}$$

Remark 6.1.3. In fact, there is a more refined relationship between $K_0(X)$ and the cohomology of X.

The Chern character of a complex vector bundle $E \to X$ is an element

$$\text{ch}(E) \in H^{2\bullet}(X; \mathbf{R}) \tag{118}$$

defined formally as follows. Suppose that ξ_i are constants and x is a formal variable such that

$$\sum_i c_i(E)x^i = \prod_i (1+\xi_i x). \tag{119}$$

Then the Chern character is defined by

$$\mathrm{ch}(E) = \sum_i e^{\xi_i}. \tag{120}$$

The Chern character enjoys a similar sum rule $\mathrm{ch}(\oplus_i E_i) = \sum_i \mathrm{ch}(E_i)$ and also a product identity

$$\mathrm{ch}(\otimes_i E_i) = \prod_i \mathrm{ch}(E_i). \tag{121}$$

Immediately, then, we see that the Chern character defines a ring homomorphism

$$\mathrm{ch}\colon K_0(X) \to H^\bullet(X). \tag{122}$$

Now, we can see how to extend Chern classes to coherent sheaves. Given a coherent sheaf $\mathcal{F}$ on a smooth projective algebraic variety over $\mathbf{C}$ there exists a locally free resolution of $\mathcal{F}$ (that is, a resolution by vector bundles) of the form

$$0 \to \mathcal{E}_{-n} \to \mathcal{E}_{-n+1} \cdots \to \mathcal{E}_{-1} \to \mathcal{E}_0 \to \mathcal{F} \to 0. \tag{123}$$

In the case of a general complex manifold such a resolution is only guaranteed to exist locally. Using such a resolution we define

$$c(\mathcal{F}) \overset{\text{def}}{=} \sum_i (-1)^i c(\mathcal{E}_i) \in H^\bullet(X). \tag{124}$$

One can show that this definition does not depend on the resolution.

This construction can be refined to providing an inverse j to the ring homomorphism $i\colon K_0(X) \to K_0(\mathrm{Coh}(X))$ by the formula

$$j([\mathcal{F}]) = \sum_i (-1)^i [\mathcal{E}_i]. \tag{125}$$

The proof of the fact that these homomorphisms are inverses to each other is outside of the scope of these notes.

An important computational tool we will use, but not spend time providing background on, is the Grothendieck–Riemann–Roch theorem. This very powerful result is a generalization of the Hirzebruch–Riemann–Roch theorem in the context of holomorphic vector bundles and complex manifolds.

Suppose that $\mathcal{E}$ is a coherent sheaf of X and that $f\colon X \to Y$ is a proper map between smooth quasi-projective varieties. The Grothendieck–Riemann–Roch theorem presents a formula for the characteristic classes of $f_!\mathcal{E}$ as

$$\mathrm{ch}(f_!\mathcal{E}) \cdot \mathrm{Td}(Y) = f_*\left(\mathrm{ch}(\mathcal{E}) \cdot \mathrm{Td}(Y)\right) \tag{126}$$

where

- ch is the Chern character as above which admits an expansion like

$$\mathrm{ch} = \mathrm{rk} + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{3!}(c_1^3 - 3c_1c_2 + 3c_3) + \cdots . \tag{127}$$

- td is the Todd class which admits an expansion like

$$\mathrm{td} = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 + \cdots \tag{128}$$

- $f_! = \sum(-1)^i \mathbf{R}^i f_*\colon K_0(X) \to K_0(Y)$ is the higher direct image or pushforward map in K-theory.
- $f_*\colon H^\bullet(X) \to H^\bullet(Y)$ is the pushforward map in cohomology.

Example 6.1.4. Suppose that $Y = pt$ and that $\mathcal{E}$ is a vector bundle $E \to X$. Then

$$\mathrm{ch}(f_!\mathcal{E}) = \chi(X, E) \tag{129}$$

is the holomorphic Euler characteristic. In this case the map f_* in cohomology is of degree $-2n$, so the theorem implies the Hirzebruch–Riemann-Roch theorem

$$\chi(X, E) = [\mathrm{ch}(E)\,\mathrm{td}(X)]_{2n}\,. \tag{130}$$

Example 6.1.5. Suppose that we have a closed embedding $i\colon Y \hookrightarrow Z$ with corresponding ideal sheaf $\mathcal{I}_Y$. From the short exact sequence

$$0 \to \mathcal{I}_Y \to \mathcal{O}_X \to \mathcal{O}_Y \to 0 \tag{131}$$

we can see that

$$c_k(\mathcal{I}_Y) = (-1)^k(k-1)![Y] \tag{132}$$

where k is the codimension of Y in X and $[Y]$ is the fundamental class of Y. In particular if X is d-dimensional and Y is zero-dimensional with n-connected components then

$$c_d(\mathcal{I}_Y) = (-1)^d n(d-1)!. \tag{133}$$

6.2. Torsion-free sheaves on surfaces

Recall that if $\mathcal{E}$ is a torsion-free sheaf on a surface X then the natural map $\mathcal{E} \to \mathcal{E}^{\vee\vee}$ is injective. In particular, there is an induced short exact sequence

$$0 \to \mathcal{E} \to \mathcal{E}^{\vee\vee} \to \mathcal{Q} \to 0 \tag{134}$$

The cokernel sheaf $\mathcal{Q}$ has the property that its support is zero-dimensional.

Consider projective space $\mathbf{P}^2$ and let $\ell_\infty \subset \mathbf{P}^2$ be the line

$$\ell_\infty = \{(0\colon z_2\colon z_3)\} \subset \mathbf{P}^2. \tag{135}$$

Definition 6.2.1. Let $\mathcal{E}$ be a torsion-free sheaf on $\mathbf{P}^2$ of rank r. A ***framing*** is an isomorphism $\Phi\colon \mathcal{E}|_{\ell_\infty} \xrightarrow{\simeq} \mathcal{O}_{\ell_\infty}^{\oplus r}$. Denote a framed sheaf by a pair $(\mathcal{E}, \Phi)$.

For a torsion-free sheaf the only topological invariant is its second Chern class, which follows from the following lemma.

Lemma 6.2.2. *Suppose $\mathcal{E}$ is a torsion-free sheaf on $\mathbf{P}^2$ which admits a framing. Then $c_1(\mathcal{E}) = 0$.*

Proof. From the short exact sequence (134) we see that $c_1(\mathcal{E}) = c_1(\mathcal{E}^{\vee\vee})$. By the framing condition, the support of the cokernel sheaf $\mathcal{Q}$ cannot intersect ℓ_∞, thus $\mathcal{E}|_{\ell_\infty} \simeq \mathcal{E}^{\vee\vee}|_{\ell_\infty}$. Finally, since $\mathcal{E}^{\vee\vee}$ is a line bundle we have $0 = c_1(\mathcal{E}^{\vee\vee}|_{\ell_\infty}) = c_1(\mathcal{E}^{\vee\vee})|_{\ell_\infty}$ which implies $c_1(\mathcal{E}^{\vee\vee}) = 0$. ☐

A map of framed sheaves

$$F\colon (\mathcal{E},\Phi) \to (\mathcal{E}',\Phi') \tag{136}$$

is a map of sheaves $F\colon \mathcal{E} \to \mathcal{E}'$ which intertwines the framings.

Definition 6.2.3. Let $\mathcal{M}^{fr}(r,n)$ be the moduli space of framed sheaves $(\mathcal{E},\Phi)$ on $\mathbf{P}^2$ of rank r and $c_2(\mathcal{E}) = n$. As a set this is the set of isomorphism classes

$$\{[(\mathcal{E},\Phi)] \mid \operatorname{rk}(E) = r, \quad c_2(\mathcal{E}) = n\}. \tag{137}$$

This only really defines $\mathcal{M}^{fr}(n,r)$ as a set, but it can be shown that it can be given the structure of a scheme [Nak99]. The rank one case is especially relevant to the previous lectures.

Proposition 6.2.4. *There is an isomorphism*

$$\mathcal{M}^{fr}(1,n) \simeq \operatorname{Hilb}_n(\mathbf{A}^2). \tag{138}$$

PROOF. Let $\mathcal{E}$ be a rank one torsion-free sheaf of second Chern class n. By the framing condition we have an embedding

$$\mathcal{E} \hookrightarrow \mathcal{E}^{\vee\vee} \simeq \mathcal{O}_{\mathbf{P}^2}. \tag{139}$$

We have already pointed out that the quotient sheaf $\mathcal{Q} = \mathcal{E}^{\vee}/\mathcal{E}$ has zero-dimensional support away from $\ell_\infty \subset \mathbf{P}^2$ and satisfies

$$\dim \Gamma(\mathbf{P}^2 - \ell_\infty, \mathcal{Q}) = n. \tag{140}$$

This gives the correspondence

$$\mathcal{M}^{fr}(1,n) \xrightarrow{\simeq} \operatorname{Hilb}_n(\mathbf{P}^2 - \ell_\infty) \simeq \operatorname{Hilb}_n(\mathbf{A}^2) \tag{141}$$

□

Next time we will explain the so-called ADHM description of the moduli space $\mathcal{M}^{fr}(r,n)$ which is in the same spirit as the description of the Hilbert scheme in terms of matrices. From there we will discuss the symplectic structure on these moduli spaces.

LECTURE 7

Moduli spaces of sheaves, III

Today we will deduce the following description of the moduli space of torsion-free rank r sheaves. Let $H(r,n)$ be the affine subspace of

$$\mathrm{End}(\mathbf{C}^n)^{\oplus n} \oplus \mathrm{Hom}(\mathbf{C}^r, \mathbf{C}^n) \oplus \mathrm{Hom}(\mathbf{C}^n, \mathbf{C}^r) \tag{142}$$

consisting of tuples (X, Y, I, J) such that

- $[X, Y] + IJ = 0$.
- there exists no proper subspace $S \subset \mathbf{C}^n$ such that $X \cdot S \subset S$, $Y \cdot S \subset S$ and $\mathrm{im}\, I \subset S$.

We refer to the last item as the stability condition. There is an action of $GL(n, \mathbf{C})$ on $H(r,n)$ defined by

$$g \cdot (X, Y, I, J) = (gXg^{-1}, gYg^{-1}, gI, Jg^{-1}). \tag{143}$$

THEOREM 7.0.1 (Barth). *Let $\mathcal{M}(r,n)$ denote the moduli space of torsion-free sheaves on $\mathbf{P}^2$ of rank r and $c_2 = n$. There is an isomorphism*

$$\mathcal{M}(r,n) \simeq H(r,n)/GL(n,\mathbf{C}). \tag{144}$$

Like the Hilbert scheme, there is a functorial definition of $\mathcal{M}(r,n)$ which makes its scheme structure manifest. The $GL(n,\mathbf{C})$ action on $H(r,n)$ is free so also like in the case of the Hilbert scheme $H(r,n)/GL(n,\mathbf{C})$ agrees with the affine GIT quotient. The above bijection of sets can be enhanced to an isomorphism of affine schemes.

This lecture closely follows the arguments in §2 of [Nak99].

7.1. Technical lemma

Suppose that $\mathcal{E}$ is any sheaf on $\mathbf{P}^2$ and let $\mathcal{E}(k) = \mathcal{E} \otimes_{\mathcal{O}} \mathcal{O}(k)$. Also let $\mathcal{Q}$ denote the rank two vector bundle

$$\mathcal{Q} = \mathcal{T}_{\mathbf{P}^2}(-1) \tag{145}$$

where $\mathcal{T}_{\mathbf{P}^2}$ is the tangent bundle. In the following we assume that $\mathcal{E}$ is a torsion-free sheaf of rank r with $c_2(\mathcal{E}) = n$. We assume that $\mathcal{E}$ is framed at the line $\ell_\infty = \{[0 : z_1 : z_2]\} \subset \mathbf{P}^2$.

Lemma 7.1.1. *For $p = 1, 2$ and $q = 0, 2$ one has*

$$H^q(\mathbf{P}^2, E(-p)) = 0 \tag{146}$$

and

$$H^q(\mathbf{P}^2, E(-1) \otimes \mathcal{Q}). \tag{147}$$

PROOF. Tensoring the short exact sequence

$$0 \to \mathcal{O}(-1) \to \mathcal{O} \to \mathcal{O}_{\ell_\infty} \to 0, \tag{148}$$

with $\mathcal{E}(-p)$ yields the short exact sequence

$$0 \to \mathcal{E}(-p-1) \to \mathcal{E}(-p) \to \mathcal{E}(-p)|_{\ell_\infty} \to 0. \tag{149}$$

Since $\mathcal{E}|_{\ell_\infty} \simeq \mathcal{O}_{\ell_\infty}^{\oplus r}$ we have

$$H^0(\mathbf{P}^2, \mathcal{E}(-p)|_{\ell_\infty}) = H^1(\mathbf{P}^2, \mathcal{E}(-p)|_{\ell_\infty}) = 0. \tag{150}$$

From the resulting long exact sequence in cohomology we obtain that

$$H^0(\mathbf{P}^2, \mathcal{E}(-p-1)) \simeq H^0(\mathbf{P}^2, \mathcal{E}(-p)) \tag{151}$$

for $p \geq 1$ and

$$H^2(\mathbf{P}^2, \mathcal{E}(-p-1)) \simeq H^2(\mathbf{P}^2, \mathcal{E}(-p)) \tag{152}$$

for $p \leq 1$.

Recall that since $\mathcal{E}$ is torsion-free we have that $\mathcal{E}^{\vee\vee}$ is locally free and that the canonical map $\mathcal{E} \hookrightarrow \mathcal{E}^{\vee\vee}$ is injective. Hence there is an injection $H^0(\mathbf{P}^2, \mathcal{E}(-p)) \hookrightarrow H^0(\mathbf{P}^2, \mathcal{E}^{\vee\vee}(-p))$. By Serre duality this means we have an inclusion

$$H^0(\mathbf{P}^2, \mathcal{E}(-p)) \hookrightarrow H^2(\mathbf{P}^2, (\mathcal{E}^{\vee\vee}(p-3)))^\vee \tag{153}$$

By the Serre vanishing theorem, for p large enough the right hand side is trivial. Combining with (151) we see that

$$H^0(\mathbf{P}^2, \mathcal{E}(-1)) \simeq H^0(\mathbf{P}^2, \mathcal{E}(-2)) \simeq \cdots \simeq 0. \tag{154}$$

Also by Serre vanishing together with (152) we see that

$$H^2(\mathbf{P}^2, \mathcal{E}(-2)) \simeq H^2(\mathbf{P}^2, \mathcal{E}(-1)) \simeq H^2(\mathbf{P}^2, \mathcal{E}) \simeq \cdots \simeq 0. \tag{155}$$

By a similar argument one obtains that $H^q(\mathbf{P}^2, \mathcal{Q} \otimes \mathcal{E}(-1))$ for $q = 0, 2$. □

7.2. THE MONADIC DESCRIPTION OF A TORSION-FREE SHEAF

Define the following vector spaces

- $V^{-1} = H^1(\mathbf{P}^2, \mathcal{E}(-2))$. Note that by the lemma above we have
$$\chi(\mathbf{P}^2, \mathcal{E}(-2)) = -\dim H^1(\mathbf{P}^2, \mathcal{E}(-2)). \tag{156}$$
Also by Hirzebruch–Riemann–Roch
$$\begin{aligned}\chi(\mathbf{P}^2, \mathcal{E}(-2)) &= \int_{\mathbf{P}^2} \mathrm{ch}(\mathcal{E}(-2))\,\mathrm{td}(\mathbf{P}^2) \\ &= r\int_{\mathbf{P}^2} \mathrm{ch}(\mathcal{O}(-2))\,\mathrm{td}(\mathbf{P}^2) - \int_{\mathbf{P}^2} c_2(E) \\ &= r\chi(\mathbf{P}^2, \mathcal{O}(-2)) - n = -n.\end{aligned}$$
Thus $\dim V^{-1} = n$, so $V^{-1} \simeq \mathbf{C}^n$.
- $V^0 = H^1(\mathbf{P}^2, \mathcal{E}(-1) \otimes \mathcal{Q}^\vee)$. This vector space has dimension $2n + r$, so $V^0 \simeq \mathbf{C}^{2n+r}$.
- $V^1 = H^1(\mathbf{P}^2, \mathcal{E}(-1))$. This vector space has dimension n, so $V^1 \simeq \mathbf{C}^n$.

We won't prove the following lemma.

Lemma 7.2.1. *Let $\mathcal{E}$ be a framed torsion-free sheaf. Then, there is a complex of sheaves*

$$(\mathcal{V}^\bullet, \mathrm{d}) \tag{157}$$

with $\mathcal{V}^i = \mathcal{O}_{\mathbf{P}^2}(i) \otimes V^i$ such that $H^{-1} = H^1 = 0$ and

$$H^0(\mathcal{V}^\bullet, \mathrm{d}) \simeq \mathcal{E}. \tag{158}$$

Let $a = \mathrm{d}_{-1\to 0}$ and $b = \mathrm{d}_{0\to 1}$ be the differentials in the complex of sheaves $\mathcal{V}^\bullet$. We can view

$$a \in H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(1)) \otimes \mathrm{Hom}(V^{-1}, V^0)$$
$$b \in H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(1)) \otimes \mathrm{Hom}(V^0, V^1)$$

In particular, a, b must be of the form

$$a = a_0 z_0 + a_1 z_1 + a_2 z_2$$
$$b = b_0 z_0 + b_1 z_1 + b_2 z_2$$

for some linear maps $a_i \colon V^{-1} \to V^0$, $b_i \colon V^0 \to V^1$. Since $\mathrm{d}^2 = 0 \iff ba = 0$ we have the relations

$$b_0 a_0 = 0, \quad b_0 a_1 + b_1 a_0 = 0$$
$$b_1 a_1 = 0, \quad b_1 a_2 + b_2 a_1 = 0$$
$$b_2 a_2 = 0, \quad b_0 a_2 + b_2 a_0 = 0.$$

By the technical lemma we can form the exact sequence

$$0 \to \mathcal{V}^{-1} \to \ker b \to \mathcal{E} \to 0. \tag{159}$$

Consider the restriction of this complex of sheaves to the line at ∞

$$0 \to \mathcal{V}^{-1}|_{\ell_\infty} \xrightarrow{a|_{\ell_\infty}} \ker b|_{\ell_\infty} \to \mathcal{E}|_{\ell_\infty} \to 0. \tag{160}$$

Note that $a|_{\ell_\infty} = a_1 z_2 + a_2 z_2$, $b|_{\ell_\infty} = b_1 z_1 + b_2 z_2$. From the resulting long exact sequence in cohomology we see that

$$H^0(\ell_\infty, \ker b|_{\ell_\infty}) \simeq H^0(\ell_\infty, \mathcal{E}|_{\ell_\infty}) \simeq \mathcal{E}|_p$$
$$H^0(\ell_\infty, \ker b|_{\ell_\infty}) \simeq H^0(\ell_\infty, \mathcal{E}|_{\ell_\infty}) = 0$$

where $p \in \ell_\infty$. The first isomorphism implies that

$$W \stackrel{\text{def}}{=} H^0(\ell_\infty, \ker b|_{\ell_\infty}) \tag{161}$$

gives the trivialization of $\mathcal{E}$ at ∞, and hence the choice of basis of W gives the framing.

Similarly, we have the short exact sequence

$$0 \to \ker b|_{\ell_\infty} \to \mathcal{V}^0|_{\ell_\infty} \xrightarrow{b|_{\ell_\infty}} \mathcal{V}^1|_{\ell_\infty} \to 0. \tag{162}$$

The long exact sequence in cohomology yields another short exact sequence

$$0 \to W \to V^0 \xrightarrow{\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}} z_1 V^1 \oplus z_2 V^1 \to 0. \tag{163}$$

where we have identified $H^0(\ell_\infty, \mathcal{O}_{\ell_\infty}(1)) \simeq \mathbf{C}z_1 + \mathbf{C}z_2$. From this short exact sequence we see that $W = \ker b_1 \cap \ker b_2$. Applying similar arguments to the dual sequence we see that the map

$$(a_1, a_2)\colon \mathcal{O}_{\ell_\infty}(1) \otimes V^{-1} \simeq z_1 V^{-1} \oplus z_2 V^{-1} \to V^0 \tag{164}$$

is injective and that $a|_{\ell_\infty}$ is injective at each fiber.

Restricting the original short exact sequence to $[0:1:0] \in \ell_\infty$ we have a sequence

$$V^{-1} \xrightarrow{a_1} V^0 \xrightarrow{b_1} V^1 \tag{165}$$

and $\ker b_1 / \operatorname{im} a_1 = E_{[0:1:0]} \simeq \ker b_1 \cap \ker b_2 = W$. Thus $\operatorname{im} a_1 \cap \ker b_2 = \operatorname{im} a_1 \cap W = 0$ so that $b_2 a_1\colon V^{-1} \to V^1$ is injective. Since V^{-1}, V^1 are of the same dimension, this map is an isomorphism.

Using this isomorphism we can identify $V = V^{-1} = V^1$ and the maps in the sequence

$$V \oplus V \xrightarrow{(a_1, a_2)} V^0 \xrightarrow{(b_1, b_2)^t} V \oplus V \tag{166}$$

with

$$a_1 = \begin{pmatrix} -\mathbb{1}_V \\ 0 \\ 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 0 \\ -\mathbb{1}_V \\ 0 \end{pmatrix} \tag{167}$$

and

$$b_1 = \begin{pmatrix} 0 & -\mathbb{1}_V & 0 \end{pmatrix}, \quad b_2 = \begin{pmatrix} \mathbb{1}_V & 0 & 0 \end{pmatrix}. \tag{168}$$

From $ba = 0$ we obtain the following form of the remaining components

$$a_0 = \begin{pmatrix} X \\ Y \\ J \end{pmatrix}, \quad b_0 = \begin{pmatrix} -Y & X & I \end{pmatrix}. \tag{169}$$

Here $[X, Y] + IJ = 0$.

This gives us the following monadic description of $\mathcal{E}$.

$$V \otimes \mathcal{O}_{\mathbf{P}^2}(-1) \xrightarrow{\quad a \quad} \begin{pmatrix} V \\ V \\ W \end{pmatrix} \otimes \mathcal{O}_{\mathbf{P}^2} \xrightarrow{\quad b \quad} V \otimes \mathcal{O}_{\mathbf{P}^2}(1). \tag{170}$$

where

$$a = \begin{pmatrix} z_0 X - z_1 \\ z_0 Y - z_2 \\ z_0 J \end{pmatrix} \tag{171}$$

and

$$b = \begin{pmatrix} -z_0 Y + z_2 & z_0 Y - z_1 & z_0 I \end{pmatrix}. \tag{172}$$

Now to go from the sheaf theoretic description to the description in terms of matrices we simply restrict this sequence to $\mathbf{P}^2 \setminus \ell_\infty \simeq \mathbf{A}^2$. The following lemma completes the result.

Lemma 7.2.2. *Suppose that* (X, Y, I, J) *satisfy* $[X, Y] + IJ = 0$. *Then*

(1) $\ker a|_{\mathbf{A}^2} = 0$.
(2) $b|_{\mathbf{A}^2}$ *is surjective if and only if the stability condition holds.*

LECTURE 8

Symplectic reduction, I

We turn to an alternative description of affine (and twisted[1]) GIT quotients. Let's motivate this by considering a simple example.

8.1. Reductive versus unitary

Let V be a vector space equipped with a hermitian metric. Let $G \subset U(V)$ be a connected closed Lie group acting by unitary transformations on V.

Warning: In this section G denotes a real compact Lie group. We will denote its complex form by $G^{\mathbf{C}}$.

We have defined the affine GIT quotient

$$V /\!/ G^{\mathbf{C}} = \operatorname{Spec} \mathbf{C}[V]^{G^{\mathbf{C}}}. \tag{173}$$

The underlying space consists of the set of closed $G^{\mathbf{C}}$-orbits. Let's consider the example

$$V = \operatorname{End}(\mathbf{C}^n) \tag{174}$$

with $g \in G = U(n)$ acting by conjugation $g \cdot B = g^{-1}Bg$. In this case, $G^{\mathbf{C}} = GL(n, \mathbf{C})$, and we have identified the closed $GL(n, \mathbf{C})$-orbits: a matrix has a closed $G^{\mathbf{C}}$-orbit if and only if it is diagonalizable. Hence

$$V /\!/ G^{\mathbf{C}} = \operatorname{End}(\mathbf{C}^n) /\!/ GL(n, \mathbf{C}) \simeq \mathbf{C}^n. \tag{175}$$

On the other hand, a matrix B satisfies

$$[B, B^\dagger] = 0 \tag{176}$$

if and only if it can be diagonalized by a unitary matrix (to see this use Schur's lemma which states that any complex square matrix is unitary equivalent to an upper triangular matrix). Thus there is a bijection

$$\operatorname{End}(\mathbf{C}^n) /\!/ GL(n, \mathbf{C}) \simeq \{B \in \operatorname{End}(\mathbf{C}^n) \mid [B, B^\dagger] = 0\}/U(n). \tag{177}$$

If we let

$$\mu \colon V \to \mathfrak{g}^* \tag{178}$$

be $\mu(B) = \frac{i}{2}[B, B^\dagger]$ (where we use the hermitian form to identify $\mathfrak{g} = Lie(U(n))$ with $\mathfrak{g}^*$) then we can rewrite this as

$$V /\!/ G^{\mathbf{C}} \simeq \mu^{-1}(0)/U(n). \tag{179}$$

[1] We introduce twisted GIT quotients today

The right hand side is called the symplectic, or Hamiltonian, reduction We will see this is a general feature about affine GIT quotients. Before that, we introduce a slight variant of the affine GIT quotient.

8.2. Twisted GIT quotient

For an affine algebraic variety X which is acted on by a reductive group G we have seen that the GIT quotient $X /\!/ G$ is in bijective correspondence with the set of closed G-orbits.

A projective variety $X \subset \mathbf{P}^n$ has a canonical $\mathbf{Z}_{\geq 0}$-grading on its algebra of global functions

$$A_\bullet = \oplus_{n \geq 0} A_n \tag{180}$$

where A_n is the algebra of degree n homogenous polynomials restricted to X. One can recover X from the graded algebra $A_\bullet$ via the 'proj' construction

$$X = \mathrm{Proj}(X). \tag{181}$$

The closed points of $\mathrm{Proj}(X)$ correspond to the set of graded ideals $J_\bullet \subset A_\bullet$ which are maximal among graded ideals not containing $A_+ = \oplus_{n>0} A_n$.

Let us return back to the situation of a reductive group G acting on an affine algebraic variety X. Suppose that χ is a character of G, meaning a homomorphism $\chi\colon G \to \mathbf{C}^\times$. Define the space of χ-twisted invariant functions to be

$$\mathbf{C}[X]^{G,\chi} \stackrel{\text{def}}{=} \{f \in \mathbf{C}[X] \mid f(g \cdot x) = \chi(g) f(x)\} \subset \mathbf{C}[X]. \tag{182}$$

Notice that when $\chi = 1$ then this return the usual G-invariants, but in general $\mathbf{C}[X]^{G,\chi}$ is not an algebra.

Even though $\mathbf{C}[X]^{G,\chi}$ is not an algebra, we obtain a canonical graded algebra by the formula

$$A_\bullet = \oplus_{n \geq 0} \mathbf{C}[X]^{G,\chi^n}. \tag{183}$$

Notice that $A_0 = \mathbf{C}[X]^G$ is the usual algebra of invariants functions.

Definition 8.2.1. The ***twisted GIT quotient*** is the quasi-projective variety

$$M /\!/_\chi G \stackrel{\text{def}}{=} \mathrm{Proj}\left(\oplus_{n \geq 0} \mathbf{C}[X]^{G,\chi^n}\right) \tag{184}$$

The canonical map $\mathbf{C}[X]^G \to \oplus_{n \geq 0} \mathbf{C}[X]^{G,\chi^n}$ induces a projective map

$$\pi\colon X /\!/_\chi G \to X /\!/ G. \tag{185}$$

Consider a character χ as above. Extend the G-action on X to a G-action on the total space of the trivial line bundle $X \times \mathbf{C}$ by the formula

$$g \cdot (x, \mu) = (g \cdot x, \chi(g^{-1})\mu). \tag{186}$$

A point $x \in X$ is called ***χ-semistable*** if for any $\mu \in \mathbf{C}^\times \subset \mathbf{C}$ the closure of the orbit of (x, μ) is disjoint from the zero section:

$$\overline{\mathcal{O}}_{(x,\mu)} \cap (X \times \{0\}) = \emptyset. \tag{187}$$

An equivalent way to understand semistability is the following.

Lemma 8.2.2. *A point $x \in X$ is χ-semistable if and only if there exists an $f \in \mathbf{C}[X]^{G,\chi^n}$, for some $n \geq 1$, such that $f(x) \neq 0$.*

From this characterization it is easy to see that the set of χ-semistable elements

$$X^{ss}_{\chi} \subset X \tag{188}$$

is G-invariant.

THEOREM 8.2.3. *Suppose that G is a reductive group acting on an affine variety X and let χ be a character of G. Then*

1. *The map $X /\!/_{\chi} G \to X /\!/ G$ is surjective.*
2. *As topological spaces one has*

$$X /\!/_{\chi} G \simeq X^{ss}_{\chi} / \sim \tag{189}$$

where $\mathbb{O} \sim \mathbb{O}'$ iff $\overline{\mathbb{O}} \cap \overline{\mathbb{O}}' \cap X^{ss} \neq \emptyset$.

3. *There is a bijection*

$$X /\!/_{\chi} G \simeq \{\text{closed orbits in } X^{ss}\}. \tag{190}$$

Notice that orbits which are closed in X^{ss} may not be closed in X, so that the twisted GIT reduction has the potential to see a wider class of orbits.

Example 8.2.4. Consider the $\mathbf{C}^{\times}$ action on $\mathbf{A}^2$ which scales each direction the same. Then we have seen that

$$\mathbf{A}^2/\mathbf{C}^{\times} = \{0\} \cup \mathbf{P}^1, \quad \mathbf{A}^2 /\!/ \mathbf{C}^{\times} = \{0\}. \tag{191}$$

Let $\chi(\lambda) = \lambda$. Then

$$(\mathbf{A}^2)^{ss}_{\chi} = \mathbf{A}^2 \setminus 0, \tag{192}$$

and hence

$$\mathbf{A}^2 /\!/_{\chi} \mathbf{C}^{\times} = \mathbf{P}^1. \tag{193}$$

LECTURE 9

Symplectic reduction, II

9.1. Resolution of singularities

Recall that quotients are the most well-behaved when the group action is free. We say a point $x \in X$ is ***regular*** if the orbit $\mathbb{O}_x$ is closed and the stabilizer G_x is trivial. The set of regular elements $X^{reg} \subset X$ is open and G-invariant. Moreover, we can consider the set of regular orbits

$$(X \mathbin{/\!/} G)^{reg} \subset X \mathbin{/\!/} G. \tag{194}$$

From the Luna slice theorem it follows that the set of regular points and the set of regular orbits $(X \mathbin{/\!/} G)^{reg}$ is open in $X \mathbin{/\!/} G$.

THEOREM 9.1.1. *The following hold.*

1. *If $x \in X^{reg}$ then x is χ-stable for any character χ.*
2. *The subspace of regular orbits $(X \mathbin{/\!/} G)^{reg}$ is nonsingular. Let $\pi\colon X \mathbin{/\!/}_\chi G \to X \mathbin{/\!/} G$ be the canonical map, then*

$$(X \mathbin{/\!/}_\chi G)^{reg} \stackrel{\text{def}}{=} \pi^{-1}(X \mathbin{/\!/} G)^{reg} \tag{195}$$

is also nonsingular and the map

$$\pi\colon (X \mathbin{/\!/}_\chi G)^{reg} \to (X \mathbin{/\!/} G)^{reg} \tag{196}$$

is an isomorphism.

Recall that a morphism $\pi\colon X \to Y$ is called a resolution of singularities if X is non-singular and π is proper and birational. Being birational means that there exists an open dense subset $Y_0 \subset Y$ such that $\pi^{-1}(Y_0)$ is dense in X and the restriction $\pi\colon \pi^{-1}(Y_0) \to Y_0$ is an isomorphism.

As a consequence of the twisted GIT theorem above we have the following result on resolutions of singularities.

Corollary 9.1.2. *Suppose that X^{reg} is nonempty and that $X \mathbin{/\!/}_\chi G$ is nonsingular and connected. Then*

$$\pi\colon X \mathbin{/\!/}_\chi G \to X \mathbin{/\!/} G \tag{197}$$

is a resolution of singularities.

PROOF. By the theorem we see that $(X \mathbin{/\!/}_\chi G)^{reg}$ is nonempty and open in $X \mathbin{/\!/}_\chi G$. □

9.2. TYPE A_1 SINGULARITY

Consider the affine subvariety

$$X \stackrel{\text{def}}{=} \{(i,j) \mid ij = 0\} \subset (\mathbf{C}^2)^* \times \mathbf{C}^2 \simeq \mathbf{C}^4. \tag{198}$$

There is an action of $\mathbf{C}^\times$ on X defined by

$$\lambda \cdot (i,j) = (\lambda i, \lambda^{-1} j). \tag{199}$$

We first consider the affine GIT quotient $X /\!/ \mathbf{C}^\times$.

Let $\mathrm{End}_0(\mathbf{C}^2) \subset \mathrm{End}(\mathbf{C}^2)$ denote the three-dimensional vector space of traceless 2×2 matrices. Every such matrix has the form

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \tag{200}$$

where $a, b, c \in \mathbf{C}$.

Lemma 9.2.1. *Let*

$$Q = \{A \mid \det(A) = 0\} \subset \mathrm{End}_0(\mathbf{C}^2) \tag{201}$$

be the singular quadric defined by the equation

$$a^2 + bc = 0 \tag{202}$$

in $\mathbf{A}^3 = \mathrm{Spec}\, \mathbf{C}[a,b,c]$. *The map*

$$\Phi \colon X /\!/ \mathbf{C}^\times \xrightarrow{\simeq} Q \tag{203}$$

defined by $\Phi(i,j) = ji$ *is an isomorphism of affine varieties.*

PROOF. It is easy to see that $\widetilde{\Phi} \colon X \to Q$ is well-defined since the condition $ij = 0$ implies that $\mathrm{tr}(ji) = \det(ji) = 0$. Clearly $\widetilde{\Phi}$ descends to the map $\Phi \colon X /\!/ \mathbf{C}^\times \to Q$. The inverse sends a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in Q \tag{204}$$

to the pair $[(i_A, j_A)]$ where $i_A = \begin{pmatrix} x & y \end{pmatrix}$ and $j_A = \begin{pmatrix} z & w \end{pmatrix}^t$. If $a = 0$ then either b or c must be zero; in the case $b = 0$ then we set $x = 0, y = 1, z = c, w = 0$. It is easy to see that this is well-defined up to the $\mathbf{C}^\times$ action. If $a \neq 0$ then set $x = a, y = b, z = 1, w = c/a$. □

Points in X with zero stabilizer are those where both i, j are nonzero. This is equivalent to the condition that $ji \neq 0$. Thus

$$(X /\!/ \mathbf{C}^\times)^{reg} \simeq Q \setminus \{0\}. \tag{205}$$

This is certainly a nonsingular variety.

Let's now consider the twisted GIT quotient. For the character let's take the identity morphism $\chi = \mathbb{1} \colon \mathbf{C}^\times \to \mathbf{C}^\times$. Suppose that $(i,j;\mu) \in X \times \mathbf{C}$ thought of as elements of the trivial line bundle over X. Then using the action in (186) we have, for $\lambda \in \mathbf{C}^\times$

$$\lambda \cdot (i,j;\mu) = (\lambda i, \lambda^{-1} j; \lambda^{-1} \mu). \tag{206}$$

Notice that if $j = 0$ and $i \neq 0, \mu \neq 0$ then

$$\mathbb{O}_{(i,0;\mu)} = \{(a, 0; \alpha) \mid a \neq 0, \alpha \neq 0\} \subset X \times \mathbf{C}. \tag{207}$$

Thus $(i, 0)$ is not a semi-stable point. Conversely we see that as long as $j \neq 0$ then the point (i, j) is semi-stable.

Proposition 9.2.2. *There is an isomorphism*

$$X /\!/_\chi \mathbf{C}^\times \simeq \mathrm{T}^*\mathbf{P}^1. \tag{208}$$

PROOF. From the characterization of semi-stable elements we see that

$$X /\!/_\chi \mathbf{C}^\times = \{(L, i) \mid L \subset \mathbf{C}^2 \text{ line}, \quad i|_L = 0\}. \tag{209}$$

Thus, there is a canonical map $X /\!/_\chi \mathbf{C}^\times \to \mathbf{P}^1$ defined by $(L, i) \mapsto L$. This map endows $X /\!/_\chi \mathbf{C}^\times$ with the structure of a line bundle over $\mathbf{P}^1$. We will identify this line bundle.

Let $L \subset \mathbf{C}^2$ be a line. A choice of a nonzero vector $v \in L$ determines an isomorphism $T_L\mathbf{P}^1 \simeq_v \mathbf{C}^2/L$. Hence

$$\mathrm{T}^*_L\mathbf{P}^1 \simeq \{i \colon \mathbf{C}^2 \to L \mid i|_L = 0\}. \tag{210}$$

This isomorphism is independent of the choice of nonzero $v \in L$. Thus $X /\!/_\chi \mathbf{C}^\times \simeq \mathrm{T}^*\mathbf{P}^1$. $\square$

From this discussion we conclude that there is a resolution of singularities

$$\pi \colon \mathrm{T}^*\mathbf{P}^1 \to Q. \tag{211}$$

This resolution is a special case of the so-called Springer resolution which we will discuss next time.

9.3. SYMPLECTIC ACTIONS

Let (M, ω) be a symplectic manifold and suppose G is acting on M.

- If M is a smooth symplectic manifold then we assume that G is a real Lie group and the action is smooth.
- If M is a symplectic algebraic variety then we assume that G is a linear algebrac group acting algebraically.

The G-action is ***symplectic*** if it preserves the symplectic form; that is for every $g \in G$ the corresponding diffeomorphism ϕ_g satisfies $\phi_g^*\omega = \omega$. Infinitesimally, this means that for every $\xi \in \mathfrak{g} = \mathrm{Lie}(G)$ the corresponding vector field $X_\xi \in \mathrm{Vect}(M)$ satisfies

$$L_{X_\xi}\omega = 0. \tag{212}$$

Such vector fields are called symplectic vector fields; the space of all symplectic vector fields $\mathrm{Vect}_\omega(M) \subset \mathrm{Vect}(M)$ is a sub Lie algebra of the Lie algebra of all smooth vector fields. So, a symplectic action ρ of G on M determines a map of Lie algebras

$$D\rho \colon \mathfrak{g} \to \mathrm{Vect}_\omega(M). \tag{213}$$

Locally, in the C^∞ world, every symplectic vector field is determined by a function. Indeed the symplectic form determines an isomorphism of $\mathrm{Vect}_\omega(M)$ with the space of *closed* one-forms $\Omega^{1,cl}(M)$. But, by the C^∞-Poincaré lemma, locally every closed one-form is exact. So, given a symplectic vector field ξ we can locally find a function $H \in C^\infty(M)$ such that

$$\xi = X_H \tag{214}$$

where $X_H = \omega^{-1}(\mathrm{d}H)$ is the Hamiltonian vector field corresponding to H.

The symplectic form ω determines a Poisson bracket $\{-,-\}$ on the commutative algebra of functions. The map

$$\{-,-\}\colon C^\infty(M) \to \mathrm{Vect}_\omega(M) \tag{215}$$

is a map of Lie algebras. Every constant function is sent to the zero vector field. Using this, one can show that there is a central extension of Lie algebras

$$0 \to \mathbf{C} \to C^\infty(M) \to \mathrm{Vect}_\omega(M) \to 0. \tag{216}$$

This extension may not be split.

Definition 9.3.1. A symplectic action ρ of G on M is ***Hamiltonian*** if there exists a G-equivariant map

$$\mu\colon M \to \mathfrak{g}^* \tag{217}$$

such that

(1) For any $a \in \mathfrak{g}$ the function

$$H_a(x) = \langle \mu(x), a \rangle \tag{218}$$

is a Hamiltonian function for the vector field $\xi_a = D\rho(a)$.

(2) The assignment $a \mapsto H_a$ is a map of Lie algebras $\mathfrak{g} \to C^\infty(M)$.

Example 9.3.2. Suppose that V is a vector space equipped with a nondegenerate skew-symmetric bilinear form $\omega \in \wedge^2 V^*$. Thus (V, ω) is a symplectic vector space. Suppose that $G \subset Sp(V)$ acts on V in a way that preserves ω. Such an action is always Hamiltonian. Indeed, define

$$\mu\colon V \to \mathfrak{g}^* \tag{219}$$

by the rule

$$\langle \mu(v), a \rangle = \frac{1}{2}\omega(v, a \cdot v), \quad \text{for all} \quad a \in \mathfrak{g}. \tag{220}$$

Here $\langle -,- \rangle$ denotes the canonical pairing between $\mathfrak{g}$ and its dual.

Example 9.3.3. Suppose that N is a smooth manifold with a G-action. Then G extends to a Hamiltonian action on T^*N with moment map defined by

$$\langle \mu(x, \eta), a \rangle_{\mathfrak{g}} = \langle \eta, \xi_a(x) \rangle \tag{221}$$

where the right-hand side is the canonical pairing between one-forms and vector fields.

A nice way to summarize the structure of the moment map is the following. We have pointed out that the algebra of functions on a symplectic manifold is equipped with a Poisson bracket. More generally, we can consider manifolds (which are not necessarily symplectic) whose functions are equipped with a Poisson bracket—such a manifold is a Poisson manifold. For any Lie algebra $\mathfrak{g}$ its dual $\mathfrak{g}^*$, thought of as a vector space, satisfies

$$\mathcal{O}(\mathfrak{g}^*) = \mathrm{Sym}(\mathfrak{g}). \tag{222}$$

The Lie bracket $[-,-]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ determines a Poisson bracket on $\mathrm{Sym}(\mathfrak{g})$. Thus, $\mathfrak{g}^*$ has the canonical structure of a Poisson manifold.

THEOREM 9.3.4. *Let M be a symplectic manifold with a Hamiltonian G action. Then the moment map $\mu\colon M \to \mathfrak{g}^*$ is a map of Poisson manifolds.*

Given a Hamiltonian G action on a symplectic manifold M, a natural question to ask is in what sense the quotient M/G is symplectic. Even if M/G is a smooth manifold it may not be the case that it is symplectic. For symplectic G-action there is a more refined procedure to produce a quotient which is symplectic (assuming it is a manifold).

LECTURE 10

Symplectic geometry, III

In this lecture we introduce the important construction of Hamiltonian reduction of a Hamiltonian action on a symplectic manifold.

10.1. Hamiltonian actions

Last time we built up towards the following definition.

Definition 10.1.1. A symplectic action ρ of G on M is ***Hamiltonian*** if there exists a G-equivariant map

$$\mu\colon M \to \mathfrak{g}^* \tag{223}$$

such that

(1) For any $a \in \mathfrak{g}$ the function

$$H_a(x) = \langle \mu(x), a \rangle \tag{224}$$

is a Hamiltonian function for the vector field $\xi_a = D\rho(a)$.

(2) The assignment $a \mapsto H_a$ is a map of Lie algebras $\mathfrak{g} \to C^\infty(M)$.

Example 10.1.2. Suppose that V is a vector space equipped with a nondegenerate skew-symmetric bilinear form $\omega \in \wedge^2 V^*$. Thus (V, ω) is a symplectic vector space. Suppose that $G \subset Sp(V)$ acts on V in a way that preserves ω. Such an action is always Hamiltonian. Indeed, define

$$\mu\colon V \to \mathfrak{g}^* \tag{225}$$

by the rule

$$\langle \mu(v), a \rangle = \frac{1}{2}\omega(v, a \cdot v), \quad \text{for all} \quad a \in \mathfrak{g}. \tag{226}$$

Here $\langle -, - \rangle$ denotes the canonical pairing between $\mathfrak{g}$ and its dual.

Example 10.1.3. Suppose that N is a smooth manifold with a G-action. Then G extends to a Hamiltonian action on T^*N with moment map defined by

$$\langle \mu(x, \eta), a \rangle_{\mathfrak{g}} = \langle \eta, \xi_a(x) \rangle \tag{227}$$

where the right-hand side is the canonical pairing between one-forms and vector fields.

Example 10.1.4. Let G be a linear algebraic group and $P \subset G$ a closed subgroup. Consider the variety $X = G/P$. There is a canonical isomorphism $\mathrm{T}_x X = g/\mathrm{Ad}_x \cdot \mathfrak{p}$ where $\mathfrak{p} = \mathrm{Lie} P$. Thus

$$\mathrm{T}^* X = \{(x, \lambda) \mid \lambda \in \mathrm{Ad}_x^* \cdot \mathfrak{p}^\perp\} \subset X \times \mathfrak{g}^* \tag{228}$$

where $\mathfrak{p}^\perp = \{\lambda \in \mathfrak{g}^* \mid \lambda|_{\mathfrak{p}} = 0\}$ and $\mathrm{Ad}^*\colon G \times \mathfrak{g}^* \to \mathfrak{g}^*$ denotes the coadjoint action.

Consider the left action of G on $X = G/P$. This extends to a Hamiltonian action of G on T^*X with moment map

$$\mu(x, \lambda) = \lambda. \tag{229}$$

A nice way to summarize the structure of the moment map is the following. We have pointed out that the algebra of functions on a symplectic manifold is equipped with a Poisson bracket. More generally, we can consider manifolds (which are not necessarily symplectic) whose functions are equipped with a Poisson bracket—such a manifold is a Poisson manifold. For any Lie algebra $\mathfrak{g}$ its dual $\mathfrak{g}^*$, thought of as a vector space, satisfies

$$\mathcal{O}(\mathfrak{g}^*) = \mathrm{Sym}(\mathfrak{g}). \tag{230}$$

The Lie bracket $[-,-]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ determines a Poisson bracket on $\mathrm{Sym}(\mathfrak{g})$. Thus, $\mathfrak{g}^*$ has the canonical structure of a Poisson manifold.

THEOREM 10.1.5. *Let M be a symplectic manifold with a Hamiltonian G action. Then the moment map $\mu\colon M \to \mathfrak{g}^*$ is a map of Poisson manifolds.*

10.2. HAMILTONIAN REDUCTION

Given a Hamiltonian G action on a symplectic manifold M, a natural question to ask is in what sense the quotient M/G is symplectic. Even if M/G is a smooth manifold it may not be the case that it is symplectic. For symplectic G-action there is a more refined procedure to produce a quotient which is symplectic (assuming it is a manifold). Let G be a real Lie group acting on a smooth manifold M.

THEOREM 10.2.1. *Suppose that M is symplectic with a proper Hamiltonian G-action with moment map $\mu\colon M \to \mathfrak{g}^*$. Let $p \in \mathfrak{g}^*$ such that*

- *p is a regular value of μ, so $\mu^{-1}(p)$ is a smooth submanifold of M.*
- *The stabilizer $G_p \subset G$ of p acts freely on $\mu^{-1}(p)$ so that $\mu^{-1}(p)/G_p = \mu^{-1}(\mathbb{O}_p)/G$ is a smooth manifold.*

Then $\mu^{-1}(p)/G_p$ has the canonical structure of a symplectic manifold compatible with the symplectic structure on M.

As a corollary we see that if G acts freely on M then $\mu^{-1}(0)/G$ has the canonical structure of a smooth symplectic manifold.

Example 10.2.2. Suppose that G acts on a smooth manifold N in a free and proper way and let $\mu\colon T^*N \to \mathfrak{g}$ be the moment map from example 10.1.3. Then there is a symplectomorphism

$$\mathrm{T}^*(N/G) \simeq \mu^{-1}(0)/G. \tag{231}$$

We now make the connection to the GIT quotient, which we recall makes sense in the affine algebro-geometric setting. Suppose that G is a reductive algebraic group acting on a nonsingular affine algebraic variety X. The cotangent bundle

T^*X is an affine algebraic variety equipped with a Hamiltonian action by G. Thus, there is an algebraic moment map $\mu\colon T^*X \to \mathfrak{g}^*$ and $\mu^{-1}(0)$ is an affine algebraic variety which is equipped with a G-action.

THEOREM 10.2.3. *Suppose the G-action on the affine algebraic variety X is free so that $X/G = X /\!/ G$ is a non-singular affine algebraic variety. Then for any G-invariant $p \in \mathfrak{g}^*$ the space $\mu^{-1}(p)/G$ is symplectic and there is a canonical symplectomorphism*

$$T^*(X/G) \simeq \mu^{-1}(0)/G. \tag{232}$$

In the above theorem we assume, importantly, that the action is free so that the GIT quotient agrees with the set-theoretic quotient. If the action is not free then generally speaking $X /\!/ G$ and $\mu^{-1}(0)/G$ are singular algebraic algebraic varieties. Nevertheless, in general $\mu^{-1}(0) \subset X$ is an affine algebraic variety and we can contemplate the GIT quotient

$$\mathcal{M}_0 \stackrel{\text{def}}{=} \mu^{-1}(0) /\!/ G. \tag{233}$$

More generally, for any character $\chi\colon G \to \mathbf{C}^\times$ we have the twisted GIT quotient

$$\mathcal{M}_\chi \stackrel{\text{def}}{=} \mu^{-1}(0) /\!/_\chi G. \tag{234}$$

By definition there is a proper map $\pi\colon \mathcal{M}_\chi \to \mathcal{M}_0$.

THEOREM 10.2.4. *Let X be a smooth affine algebraic variety and let G be a reductive algebra group acting on X. Then*

1. *For any character χ the variety $\mathcal{M}_\chi$ is Poisson. The morphism $\pi\colon \mathcal{M}_\chi \to \mathcal{M}_0$ is Poisson.*
2. *Let X^s be the χ-stable points so that $X^s /\!/ X /\!/_\chi G$ is a smooth subvariety. Also*

$$\mathcal{M}^s_\chi \stackrel{\text{def}}{=} (\mu^{-1}(0)^s) /\!/ G \subset \mathcal{M}_\chi \tag{235}$$

is smooth. Then $\mathcal{M}^s_\chi$ is symplectic and contains $T^*(X^s /\!/ G)$.

LECTURE 11

The Springer resolution

Today we will consider an example of a symplectic resolution of singularities. Recall that we are in the following situation. We have a Hamiltonian action of a reductive group G on a symplectic variety X. We can consider the following flavors of Hamiltonian reduction of the Hamiltonian G action on the cotangent bundle T^*X:

$$\mathcal{M}_0 = \mu^{-1}(0) /\!/ G \tag{236}$$

where $\mu\colon X \to \mathfrak{g}^*$ is the moment map and its twisted version

$$\mathcal{M}_\chi = \mu^{-1}(0) /\!/_\chi G \tag{237}$$

where $\chi\colon G \to \mathbf{C}^\times$ is a character. There is a canonical map

$$\pi\colon \mathcal{M}_\chi \to \mathcal{M}_0. \tag{238}$$

By the result we stated last time we see that when $\mathcal{M}_\chi$ is non-singular then this map is a resolution of singularities with the additional conditions that

- $\mathcal{M}_\chi$ is symplectic and
- $\mathcal{M}_0$ is Poisson and the map π is a Poisson map.

We call such a resolution of singularities a *symplectic resolution* of singularities.

11.1. A REMINDER OF THE A_1 CASE

Consider $X = \mathbf{A}^2$ with its standard $G = \mathbf{C}^\times$ action by scaling. Then $\mathbf{C}^\times$ acts on the symplectic variety

$$T^*X = \{(i,j) \mid i \in (\mathbf{C}^2)^*, j \in \mathbf{C}^2\} \tag{239}$$

in a Hamiltonian way with moment map $\mu(i,j) = ij$.

We have seen that the affine GIT reduction of $\mu^{-1}(0)$ by $\mathbf{C}^\times$ is the following quadric

$$\mathcal{M}_0 \overset{\text{def}}{=} \mu^{-1}(0) /\!/ \mathbf{C}^\times \simeq Q = \{(a,b,c) \mid a^2 + bc = 0\} \tag{240}$$

Also, for $\chi(\lambda) = \lambda$ we have seen that the twisted GIT reduction is

$$\mathcal{M}_\chi = \mu^{-1}(0) /\!/_\chi \mathbf{C}^\times \simeq T^*\mathbf{P}^1. \tag{241}$$

Furthermore, we have the resolution of singularities

$$\pi\colon T^*\mathbf{P}^1 \to Q. \tag{242}$$

Of course, $T^*\mathbf{P}^1$ is naturally a symplectic manifold. There is also a Poisson structure on Q defined by

$$\{b,c\} = 2a, \quad \{a,b\} = b, \quad \{a,c\} = -c. \tag{243}$$

Each of these structures is compatible with the standard symplectic structure on $T^*\mathbf{A}^2$, and furthermore the map π is a Poisson morphism.

11.2. The Springer resolution

We consider a generalization of this example. In the remainder of this section we take $G = SL(n,\mathbf{C})$, but any semi-simple reductive group will work. Consider the ***nilpotent cone*** defined by

$$\mathcal{N} \stackrel{\text{def}}{=} \{a \in \mathfrak{g} \mid a^N = 0, \quad \text{for some } N\} \subset \mathfrak{g}. \tag{244}$$

Then $\mathcal{N}$ is an affine algebraic variety and it is equipped with a $\mathbf{C}^\times$ action $a \mapsto \lambda a$ (this is why it is called a 'cone'). When $G = SL(2,\mathbf{C})$ then we have $\mathcal{N} = Q$ from the previous example. In general, $\mathcal{N}$ is a singular affine variety with cone point $0 \in \mathcal{N}$.

Consider a maximal torus $T \subset G$ with Lie algebra $\mathfrak{h}$. Let $B \subset G$ be a Borel subgroup containing T. In the case $G = SL(n,\mathbf{C})$ we can take B to be the subgroup of upper triangular matrices. Define the ***flag variety*** of G to be

$$\mathcal{F} \stackrel{\text{def}}{=} G/B \tag{245}$$

It is isomorphic to the variety of all Borel subgroups of G. In the case $G = SL(n,\mathbf{C})$ this is isomorphic to the set of full flags of n-dimensional space

$$0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n \simeq \mathbf{C}^n \tag{246}$$

where $\dim V_i = i$. We will denote such a flag by $V_\bullet \in \mathcal{F}$.

Consider the special case of $G = SL(n,\mathbf{C})$. Let

$$\widetilde{\mathcal{N}} \stackrel{\text{def}}{=} \{(V_\bullet, y) \mid yV_i \subset V_{i-1}\} \subset \mathcal{F} \times \mathfrak{sl}(n,\mathbf{C}). \tag{247}$$

Notice that the condition $yV_i \subset V_{i-1}$ implies that y is nilpotent, thus

$$\widetilde{\mathcal{N}} \subset \mathcal{F} \times \mathcal{N}. \tag{248}$$

If we think about $\mathcal{F}$ instead as the space of all Borel subalgebras then $\widetilde{\mathcal{N}}$ is the set of pairs $(\mathfrak{b}, y)$ such that $y \in \mathfrak{b}$. The projection $\widetilde{\mathcal{N}} \to \mathcal{F}$ is a vector bundle with fiber $[\mathfrak{b},\mathfrak{b}] = \mathfrak{n}$.

Proposition 11.2.1. *There is an isomorphism $\widetilde{\mathcal{N}} \simeq T^*\mathcal{F}$. In particular, $\widetilde{\mathcal{N}}$ carries a symplectic structure. Further, the natural left action of $SL(n,\mathbf{C})$ on $\widetilde{\mathcal{N}}$ is Hamiltonian with moment map*

$$\mu\colon \widetilde{\mathcal{N}} \to \mathfrak{sl}(n,\mathbf{C}) \tag{249}$$

defined by $\mu(V,y) = y$.

Proof. Fix a basis $\{e_1,\ldots,e_n\}$ on $\mathbf{C}^n$ and let $V^0_\bullet$ denote the standard flag given by

$$V^0_i = \text{span}\{e_1,\ldots,e_i\}. \tag{250}$$

Then the Borel subgroup B is

$$B = \{g \in SL(n, \mathbf{C}) \mid g \cdot V_i^0 \subset V_i\}. \tag{251}$$

Now, from example 10.1.4 we have an identification

$$\mathrm{T}^*\mathcal{F} = \{(V, y) \mid \operatorname{tr}(ay) = 0, \text{for any } a \text{ with } a \cdot V_i \subset V_i\} \subset \mathcal{F} \times \mathfrak{sl}(n, \mathbf{C}). \tag{252}$$

This implies that $\widetilde{\mathcal{N}} \simeq \mathrm{T}^*\mathcal{F}$ as desired. □

Since the image of the moment map μ is contained in the nilpotent elements we see that it defines a map

$$\mu \colon \widetilde{\mathcal{N}} \to \mathcal{N}. \tag{253}$$

It turns out that this is a symplectic resolution of singularities. For more details we refer to [CG10].

11.3. Springer resolution in general

We briefly sketch how this construction is generalized to the case of an arbitrary semisimple group G.

Lemma 11.3.1. *There is an isomorphism*

$$\mathrm{T}(\mathcal{F}) \simeq G \times^B \mathfrak{g}/\mathfrak{b} \tag{254}$$

Proof. Consider the trivial vector bundle on the G/B with fiber $\mathfrak{g}$. There is a surjective map of vector bundles

$$L \colon G/B \times \mathfrak{g} \to \mathrm{T}(G/B) \tag{255}$$

which sends a pair (gB, x) to the pair $(gB, \xi_x(gB))$ where ξ_x is the vector field on G/B determined by the infinitesimal left action of $x \in \mathfrak{g}$. The kernel of this map is the vector bundle whose fiber over gB is the Lie algebra of the stabilizer of gB; which is the Borel subalgebra $\mathrm{Ad}_g\mathfrak{b}$. There is an isomorphism

$$G \times^B \mathfrak{b} \to \ker L \tag{256}$$

sending $[g, x] \mapsto (gB, gxg^{-1})$. Thus we have an isomorphism of vector bundles

$$\mathrm{T}(G/B) \simeq (G/B \times \mathfrak{g})/(G \times^B \mathfrak{b}) \simeq G \times^B \mathfrak{g}/\mathfrak{b} \tag{257}$$

as desired. □

As a consequence we have

$$\mathrm{T}^*(G/B) \simeq G \times^B (\mathfrak{g}/\mathfrak{b})^*. \tag{258}$$

Since G is semisimple we identify this with

$$\mathrm{T}^*(G/B) \simeq G \times^B \mathfrak{b}^\perp, \tag{259}$$

where $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{b}^\perp$ is the orthogonal decomposition with respect to the Killing form. Furthermore, there is a direct sum decomposition $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ where $\mathfrak{n} \subset \mathfrak{b}$ is the subalgebra of nilpotent elements in $\mathfrak{b}$. One can show that $\mathfrak{n} \subset \mathfrak{b}^\perp$, so that $\mathrm{T}^*(G/B) \simeq G \times^B \mathfrak{n}$.

Now, G acts on G/B so it automatically acts on $\mathrm{T}^*(G/B)$ in a Hamiltonian way. Explicitly, the moment map

$$\mu\colon G \times^B \mathfrak{n} \to \mathfrak{g}^* \simeq \mathfrak{g} \tag{260}$$

is $\mu([g,x]) = gxg^{-1}$. Again, it is immediate to see that this map lands in the set of nilpotent elements $\mathcal{N} \subset \mathfrak{g}$, so it defines a map

$$\mu\colon \mathrm{T}^*(G/B) \to \mathcal{N}. \tag{261}$$

Again, this is a symplectic resolution of singularities.

11.4. Kähler quotients

We return to the relationship between the quotient of a vector space by a real compact Lie group and the GIT reduction by the corresponding complex reductive group. Let K be a compact real Lie group and let G be its complex form. We assume that $K \subset G$ and that the complexification $K(\mathbf{C}) \simeq G$. Notice that at the level of Lie algebras we have $\mathfrak{g} = \mathfrak{k} \otimes_{\mathbf{R}} \mathbf{C}$.

Let $V_{\mathbf{R}}$ be a real vector space equipped with a positive definite inner product g and a compatible complex structure I. When we chose to view $V_{\mathbf{R}}$ as a complex vector space via I we denote it by V. This equips $V_{\mathbf{R}}$ with the structure of a Kähler manifold. Let $(-,-)$ be the corresponding Hermitian inner product and let $\omega = \mathrm{Im}(-,-) \in \wedge^2 V_{\mathbf{R}}^*$ be the symplectic form. Suppose that $K \subset U(V_{\mathbf{R}})$ acts unitarily on V and hence its complexification $G \subset GL(V,\mathbf{C})$ acts through complex linear transformations.

By assumption, the action of K is symplectic with respect to ω. Thus we have a moment map

$$\mu_{\mathbf{R}}\colon V_{\mathbf{R}} \to \mathfrak{k}^* \tag{262}$$

defined explicitly by the rule that

$$\langle \mu_{\mathbf{R}}(v), a\rangle = \frac{1}{2}\omega(x, a\cdot x). \tag{263}$$

But, since ω is the imaginary part of the Hermitian inner product, we can write the moment map as $\langle \mu_{\mathbf{R}}(v), a\rangle = \frac{i}{2}(a\cdot x, x)$.

In this situation we can contemplate two quotients, namely $\mu_{\mathbf{R}}^{-1}(0)/K$ and $V /\!/ G$. (Recall that generally the G-orbit of an arbitrary element $v \in V$ will not be G-closed, so there is no expected relationship between V/K and $V /\!/ G$.) The Kempf–Ness theorem states that these quotients can naturally be identified.

Theorem 11.4.1 ([KN79]). *The G-orbit of any $x \in \mu_{\mathbf{R}}^{-1}(0)$ in V is closed and furthermore there is an isomorphism*

$$\mu_{\mathbf{R}}^{-1}(0)/K \xrightarrow{\simeq} V /\!/ G \tag{264}$$

intertwining the complex structures.

We will not prove this theorem, but let us unpack the last statement. By definition, the space $\mu^{-1}(0)/K$ is the real Hamiltonian reduction of V by the K-action. It inherits a Kähler structure from that on V. On the other hand, the right hand side $V /\!/ G$ is the affine GIT quotient, which is a complex affine variety by definition.

LECTURE 12

The Hilbert scheme as a reduction

Recall the following example. Equip $V = \mathrm{End}(\mathbf{C}^n)$ with a Kähler structure induced from the standard one on $\mathbf{C}^n$. The Hermitian inner product is simply $(x, y) = \mathrm{tr}(xy^\dagger)$ and the symplectic form is $\omega(x, y) = \mathrm{Im}(x, y)$. Consider the adjoint action through unitary matrices

(265) $$\mathrm{End}(\mathbf{C}^n) \ni x \mapsto g^{-1}xg, \quad g \in U(n).$$

The corresponding moment map is simply

(266) $$\mu_{\mathbf{R}}(x) = \frac{\mathrm{i}}{2}[x, x^\dagger].$$

The Kempf–Ness theorem gives a natural isomorphism

(267) $$\mu^{-1}(0)/U(n) \simeq \mathrm{End}(\mathbf{C}^n) /\!/ GL(n, \mathbf{C}).$$

We have seen that the right hand side is isomorphic to $\mathbf{C}^n$; the closed orbits are the diagonalizable matrices and $\mathbf{C}^n$ consists of the sets of eigenvalues. The right hand side is the quotient by $U(n)$ of matrices satisfying $[x, x^\dagger] = 0$. Any normal matrix can be diagonalized by a unitary matrix. This is an explicit example of the Kempf–Ness theorem.

12.1. THE SYMMETRIC PRODUCT

Let V, W be complex vector spaces of dimension n and 1 respectively. Let

(268) $$H_{n,1} \overset{\mathrm{def}}{=} \mathrm{End}(V) \oplus \mathrm{End}(V) \oplus \mathrm{Hom}(W, V) \oplus \mathrm{Hom}(V, W).$$

There is a natural $GL(V, \mathbf{C})$ action on $H_{n,1}$ defined by

(269) $$(B_1, B_2, i, j) \mapsto (g^{-1}B_1g, g^{-1}B_2g, g^{-1}i, jg), \quad g \in GL(V, \mathbf{C}).$$

Notice that

(270) $$H_{n,1} = \mathrm{T}^* \mathrm{End}(V) \oplus \mathrm{T}^* \mathrm{Hom}(W, V).$$

In particular, $H_{n,1}$ is naturally a complex symplectic vector space. The $GL(V, \mathbf{C})$ action is Hamiltonian with respect to this symplectic structure and the (holomorphic) moment map

(271) $$\mu_{\mathbf{C}} \colon H_{n,1} \to \mathfrak{gl}(V)^* \simeq \mathfrak{gl}(V)$$

is

(272) $$\mu_{\mathbf{C}}(B_1, B_2, i.j) = [B_1, B_2] + ij.$$

By our work in previous lectures we have identified the nth symmetric product of $\mathbf{C}^2$ with the GIT Hamiltonian reduction

$$S^n\mathbf{C}^2 \simeq \mu_{\mathbf{C}}^{-1}(0) /\!/ GL(V,\mathbf{C}). \tag{273}$$

Via this description it makes it manifest that $S^n\mathbf{C}^n$ is equipped with a Poisson structure. (Describe it explicitly.)

Equip V, W with Hermitian inner products so that the vector space is equipped with an induced Hermitian inner product. We have a natural action of $g \in U(V) \simeq U(n)$ on $H_{n,1}$ defined by restriction of the action (269). The real moment map for this unitary action is

$$\mu_1(B_1, B_2, i, j) = \frac{\mathrm{i}}{2}\left([B_1, B_1^\dagger] + [B_2, B_2^\dagger] + ii^\dagger - j^\dagger j\right). \tag{274}$$

Thus, by the Kempf–Ness theorem we have another description of $S^n\mathbf{C}^2$

$$\mu_{\mathbf{C}}^{-1}(0) /\!/ GL(n,\mathbf{C}) \simeq S^n\mathbf{C}^2 \simeq \left(\mu_1^{-1}(0) \cap \mu_{\mathbf{C}}^{-1}(0)\right)/U(n). \tag{275}$$

12.2. Hilbert scheme

Consider the complex vector space $H_{n,1}$ equipped with its $GL(n,\mathbf{C})$ action. Define the character $\chi\colon GL(n,\mathbf{C}) \to \mathbf{C}^\times$ by

$$\chi(g) = (\det g)^l \tag{276}$$

where l is an arbitrary positive integer.

Proposition 12.2.1. *There is an isomorphism*

$$\mathrm{Hilb}_n(\mathbf{C}^2) \simeq \mu_{\mathbf{C}}^{-1}(0) /\!/_{\chi} GL(n). \tag{277}$$

This result follows from the following lemma asserting that the familiar stability condition that we originally used in the description of the Hilbert scheme translates to the statement that orbits are closed in the semi-stable locus.

Lemma 12.2.2. *The tuple (B_1, B_2, i, j) satisfies the stability condition if and only if it is χ-semi stable.*

Proof. Recall that the stability condition says that there is no subspace $S \subset V$ such that

- S is invariant for B_1, B_2.
- $\mathrm{im}(i) \subset S$.

By way of contradiction let's assume that there exists such an S and that

$$G \cdot (B_\alpha, i, j; z) \subset H_{n,1} \times \mathbf{C} \tag{278}$$

is closed.

As we have done before, let's take a complementary subspace $S^\perp$ such that $V = S \oplus S^\perp$. Then in this form the matrices B_α take the form

$$B_\alpha = \begin{pmatrix} \star & \star \\ 0 & \star \end{pmatrix} \tag{279}$$

And i is a column vector of the form $i = \begin{pmatrix} \star & 0 \end{pmatrix}^t$. Let

(280) $$g(t) \stackrel{\text{def}}{=} \begin{pmatrix} \mathbb{1}_S & 0 \\ 0 & t^{-1}\mathbb{1}_{S^\perp} \end{pmatrix}.$$

Then

(281) $$g(t)B_\alpha g(t)^{-1} = \begin{pmatrix} \star & t\star \\ 0 & \star \end{pmatrix}, \quad g(t)i = i.$$

On the other hand $(\det g(t))^{-l}z = t^{l\cdot \dim S^\perp}z \to 0$ as $t \to 0$ since $\dim S^\perp > 0$ by assumption. This contradicts the fact that $G \cdot (B_\alpha, i, j; z)$ is closed.

Next suppose that the stability condition holds. By contradiction suppose that $G \cdot (B_\alpha, i, j; z)$ is not closed.

□

To get a similar description of the Hilbert scheme as a Kähler quotient we need to discuss a small generalization of the Kempf–Ness theorem where the affine GIT quotient is replaced by the twisted GIT quotient.

Suppose K is a compact Lie group with complexification G both acting in an appropriate way on a Hermitian vector space V. Let $\chi\colon G \to \mathbf{C}^\times$ be a character which restricts to a character $\chi_{\mathbf{R}}\colon K \to U(1)$. We identify $\mathfrak{u}(1) \simeq \mathrm{i}\mathbf{R}$. Then, the variant of the Kempf–Ness theorem is an isomorphism

(282) $$\mu_{\mathbf{R}}^{-1}(\mathrm{i}\, d\chi_{\mathbf{R}})/K \simeq V /\!/_\chi G.$$

Applied to the Hilbert scheme example we then have a sequence of isomorphisms

(283) $$\mu_{\mathbf{C}}^{-1}(0) /\!/_\chi GL(n, \mathbf{C}) \simeq \mathrm{Hilb}_n(\mathbf{C}^2) \simeq \left(\mu_1^{-1}(\mathrm{id}\chi_{\mathbf{R}}) \cap \mu_{\mathbf{C}}^{-1}(0)\right) /U(n).$$

12.3. HyperKähler quotients

In this section we will survey the result that the Hilbert scheme on $\mathbf{C}^2$, and more generally the moduli of torsion-free sheaves, can be given the structure of a hyperkähler manifold.

Recall that a Kähler manifold is a Riemannian manifold of dimension $2n$ with a compatible almost complex structure I which is integrable and such that the Kähler two-form ω is d-closed. This is equivalent to asking that the complex structure I be parallel with respect to the Levi-Civita connection $\nabla I = 0$. For a Kähler manifold, the holonomy group of ∇ is contained in $U(n)$. In other words, the $SO(2n)$ bundle of frames admits a reduction of structure to $U(n)$.

A hyperKähler manifold is a smooth Riemannian manifold (M, g) with a triple of almost complex structures I, J, K satisfying

(1) Each I, J, K preserve the metric g.
(2) I, J, K satisfy the quaternionic relations $I^2 = J^2 = K^2 = IJK = -1$.
(3) I, J, K are parallel with respect to the Levi-Civita connection $\nabla I = \nabla J = \nabla K = 0$.

These conditions imply that the holonomy group of ∇ is contained in the real symplectic group $Sp(n) \subset SO(4n)$. Each pair $(g, I), (g, J), (g, K)$ defines a Kähler structure with Kähler forms we denote by $\omega_I, \omega_J, \omega_K$. If we fix the complex structure I then the combination

$$\omega_{\mathbf{C}} \stackrel{\text{def}}{=} \omega_J + \mathrm{i}\omega_K \tag{284}$$

is holomorphic. Meaning $\omega_{\mathbf{C}}$ is Hodge type $(2,0)$ and is $\overline{\partial}_I$-closed.

Suppose that K is a compact real Lie group acting on a hyperKähler manifold X in a way that preserves I, J, K, g.

Definition 12.3.1. A map

$$\mu\colon X \to \mathbf{R}^3 \otimes \mathfrak{k}^* \tag{285}$$

is a ***hyperKähler moment map*** if

(1) μ is K-equivariant.
(2) If $\mu = (\mu_I, \mu_J, \mu_K)$ then

$$\langle \mathrm{d}\mu_I(v), a\rangle = \omega_I(\xi_a, v) \tag{286}$$

for any $v \in \mathrm{T}X$, $a \in \mathfrak{k}$ and similarly for J, K.

Suppose that X is equipped with such a moment map.

THEOREM 12.3.2 (**[Hitchin]**). *Suppose $\zeta_1, \zeta_2, \zeta_3 \in \mathfrak{k}^*$ are Ad-invariant elements. Then if $\zeta = (\zeta_1, \zeta_2, \zeta_3)$ the set $\mu^{-1}(\zeta) \subset X$ is K-invariant.*

If we assume that the K-action on $\mu^{-1}(\zeta)$ is free then the quotient space $\mu^{-1}(\zeta)/K$ is a smooth manifold equipped with a hyperKähler structure compatible with the one on X.

The resulting space $\mu^{-1}(\zeta)/K$ is called the hyperKähler quotient and is sometimes denoted

$$X /\!\!/\!\!/ K \stackrel{\text{def}}{=} \mu^{-1}(\zeta)/K. \tag{287}$$

LECTURE 13

Hyperkähler reduction

Last time we introduced the notion of hyperkähler reduction. Given a hyperkähler manifold X and a hyperkähler moment map $\mu\colon X \to \mathrm{Lie}(K)^* \otimes \mathbf{R}^3$ the hyperkähler reduction is

$$X /\!\!/\!\!/ K = \mu^{-1}(\zeta)/K \tag{288}$$

where ζ is a K-invariant element. For this quotient to be smooth hyperkähler manifold we need to assume that the K-action on $\mu^{-1}(\zeta)$ is free.

13.1. The Hilbert scheme as a hyperkähler quotient

Let

$$H_{n,1} \stackrel{\text{def}}{=} \mathrm{End}(V) \oplus \mathrm{End}(V) \oplus \mathrm{Hom}(W,V) \oplus \mathrm{Hom}(V,W). \tag{289}$$

be the space of tuples (B_α, i, j) as usual where V, W are hermitian vector spaces of dimension n and 1 respectively.

Lemma 13.1.1. *Suppose that H is any hermitian vector space. Suppose that $J\colon H \to H$ is a real endomorphism satisfying*

- *J is anti-linear meaning $J(\lambda v) = \overline{\lambda} J(v)$.*
- *$J^2 = -\mathbb{1}$.*

Then J endows H with the structure of a quaternionic vector space.

Proof. Let $I\colon H \to H$, $v \mapsto \mathrm{i}v$ be the complex structure underlying V. Define $K\colon H \to H$ by $K = IJ$. In other words, $K(v) = \mathrm{i}J(v)$. Clearly $KK(v) = \mathrm{i}J(\mathrm{i}J(v)) = J^2(v) = -v$ so that $K^2 = -\mathbb{1}$. Also $JK(v) = J(\mathrm{i}J(v)) = -\mathrm{i}J^2(v) = \mathrm{i}v$ so that $JK = I = -KJ$. The remaining relation $KI = J = -IK$ is similar to check. □

As a corollary we see that $H_{n,1}$ is equipped with a quaternionic structure.

Corollary 13.1.2. *The anti-linear endomorphism $J\colon H_{n,1} \to H_{n,1}$ defined by*

$$J(B_1, B_2, i, j) = (B_2^\dagger, -B_1^\dagger, j^\dagger, -i^\dagger) \tag{290}$$

endows $H_{n,1}$ with the structure of a quaternionic vector space.

Recall the holomorphic moment map

$$\mu_{\mathbf{C}}\colon H_{n,1} \to \mathfrak{gl}(V) = \mathfrak{gl}(n) \tag{291}$$

defined by $\mu_{\mathbf{C}}(B_\alpha, i, j) = [B_1, B_2] + ij$. Decompose this into its real and imaginary parts

$$\mu_{\mathbf{C}} = \mu_J + \mathrm{i}\mu_K \tag{292}$$

where we view $\mathfrak{gl}(n, \mathbf{C})$ as the complexification of $\mathfrak{u}(n)$. Also set $\mu_I = \mu_1$ so that

$$\mu_I(B_1, B_2, i, j) = \frac{\mathrm{i}}{2}\left([B_1, B_1^\dagger] + [B_2, B_2^\dagger] + ii^\dagger - j^\dagger j\right). \tag{293}$$

It is easy to see that $\mu = (\mu_I, \mu_J, \mu_K)$ is a hyperkähler moment map for the $U(n)$ action on $H_{n,1}$. The following is an immediate consequence of the theorem from last class.

THEOREM 13.1.3. *Let $\chi(g) = (\det g)^l$ be the real character used in the last class. Then the $U(n)$ action on*

$$\zeta_\chi = \mu^{-1}(\mathrm{id}\chi, 0, 0) \tag{294}$$

is free. In particular the Hilbert scheme of n points in $\mathbf{A}^2$

$$\mathrm{Hilb}_n(\mathbf{A}^2) \simeq H_{n,1} /\!/\!/ U(n) = \mu^{-1}(\zeta_\chi)/U(n) \tag{295}$$

is a hyperkähler quotient.

Since $\mu = (\mu_I, \mu_J, \mu_K)$ we can rewrite this as

$$\mathrm{Hilb}_n(\mathbf{A}^2) \simeq \mu_I(-\mathrm{id}\chi) \cap \mu_J(0) \cap \mu_K(0)/U(n). \tag{296}$$

13.2. CALOGERO–MOSER SPACES

For any $\zeta \in \mathbf{R}^3 \otimes \mathfrak{u}(n)$ we can consider the quotient

$$H_{n,1} /\!/\!/_\zeta U(n) \stackrel{\text{def}}{=} \mu^{-1}(\zeta_\chi)/U(n). \tag{297}$$

If we take $\zeta = 0$ we recover the symmetric product $S^n\mathbf{A}^2$. As long as $\zeta \neq 0$ the $U(n)$ action on $\mu^{-1}(\zeta_\chi)$ is free so that $H_{n,1} /\!/\!/_\zeta U(n)$ is a smooth hyperkähler manifold.

For $\zeta \neq 0$ the spaces $H_{n,1} /\!/\!/_\zeta U(n)$ are related through hyperkähler rotation. Precisely, there exists a matrix $R \in SO(3)$ which acts on the hyperkähler structure as

$$\begin{pmatrix} I' \\ J' \\ K' \end{pmatrix} = R \begin{pmatrix} I \\ J \\ K \end{pmatrix} \tag{298}$$

which satisfies

$$R\zeta = \begin{pmatrix} |\zeta| \\ 0 \\ 0 \end{pmatrix}. \tag{299}$$

This transformation is compatible with the hyperkähler quotient and so descends to an isometry

$$R\colon H_{n,1} /\!/\!/_\zeta U(n) \xrightarrow{\simeq} \mathrm{Hilb}_n(\mathbf{A}^2). \tag{300}$$

So, as $\zeta \neq 0$ varies all of the spaces $H_{n,1} /\!\!/\!\!/_\zeta U(n)$ are isometric. But, they are not the same as complex manifolds; indeed the hyperkähler rotation moves around the complex structures. Choose an identification

$$\mathbf{R}^3 = \mathbf{R} \oplus \mathbf{C} \tag{301}$$

and decompose $\zeta = (\zeta_{\mathbf{R}}, \zeta_{\mathbf{C}})$ and $\mu = (\mu_{\mathbf{R}}, \mu_{\mathbf{C}})$. Then, following (296) we can write

$$\mathrm{Hilb}_n(\mathbf{A}^2) = \mu_{\mathbf{R}}^{-1}(\mathbb{1}) \cap \mu_{\mathbf{C}}^{-1}(0)/U(n). \tag{302}$$

One can show that as long as $\zeta_{\mathbf{C}} \neq 0$ that

- $\mu_{\mathbf{C}}^{-1}(\zeta_{\mathbf{C}})$ is non-singular.
- Every $GL(n, \mathbf{C})$-orbit in $\mu_{\mathbf{C}}^{-1}(\zeta_{\mathbf{C}})$ is closed.
- The stabilizer of every point in $\mu^{-1}(\zeta)$ is trivial.

For such $\zeta = (\zeta_{\mathbf{R}}, \zeta_{\mathbf{C}})$ we see that $\mu_{\mathbf{C}}^{-1}(\zeta_{\mathbf{C}})/GL(n, \mathbf{C})$ agrees with the affine GIT quotient $\mu_{\mathbf{C}}^{-1}(\zeta) /\!\!/ GL(n, \mathbf{C})$. Furthermore, by Kempf–Ness we see that

$$H_{n,1} /\!\!/_\zeta U(n) = \mu^{-1}(\zeta)/U(n) \simeq \mu_{\mathbf{C}}^{-1}(\zeta_{\mathbf{C}}) /\!\!/ GL(n, \mathbf{C}). \tag{303}$$

Combining these facts with the isometry above, we see that $\mathrm{Hilb}_n(\mathbf{A}^2)$ is diffeomorphic to an affine algebraic manifold. More explicitly, the ***Calogero–Moser*** space is defined to be

$$\mathcal{CM}_n \stackrel{\text{def}}{=} \mu_{\mathbf{R}}^{-1}(0) \cap \mu_{\mathbf{C}}^{-1}(-\mathbb{1})/U(n). \tag{304}$$

Thus, $\mathcal{CM}_n$ is the hyperkähler reduction $H_{n,1} /\!\!/\!\!/_\zeta U(n)$ where $\zeta = (0, -\mathbb{1})$ and $\mathrm{Hilb}_n(\mathbf{A}^2)$ is the hyperkähler reduction $H_{n,1} /\!\!/\!\!/_\zeta U(n)$ where $\zeta = (\mathbb{1}, 0)$.

The hyperkähler rotation of $\zeta = (0, -\mathbb{1})$ into $\zeta = (\mathbb{1}, 0)$ corresponds to multiplication by the quaternion $(\mathrm{i} + \mathrm{k})/\sqrt{2}$. In other words, the real endomorphism

$$\frac{1}{\sqrt{2}}(\mathrm{i} + \mathrm{k}) \colon H_{n,1} \to H_{n,1} \tag{305}$$

is $U(n)$ equivariant and induces a real isometry

$$\mathcal{CM}_n \xrightarrow{\simeq} \mathrm{Hilb}_n(\mathbf{A}^2). \tag{306}$$

The Calogero–Moser space $\mathcal{CM}_n$ can thus be understood as the holomorphic symplectic reduction

$$\mu_{\mathbf{C}}^{-1}(-\mathbb{1})/GL(n, \mathbf{C}) = \{(B_1, B_2, i, j) \mid [B_1, B_2] + ij = \mathbb{1}\}/GL(n, \mathbf{C}). \tag{307}$$

There are special Hamiltonian flows on the space $\mathcal{CM}_n$ induced by the Hamiltonians

$$H_k = (-1)^{k-1} \operatorname{tr} B_2^k \in \mathbf{C}[H_{n,1}]. \tag{308}$$

The functions H_k are $GL(n, \mathbf{C})$-invariant and hence determine Hamiltonians

$$H_k \in \mathcal{O}(\mathcal{CM}_n), \quad k = 1, \ldots, n. \tag{309}$$

These functions satisfy

$$\{H_i, H_j\} = 0, \quad \forall i, j \tag{310}$$

and comprise the famous Calogero–Moser integrable system.

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