LECTURE 3

Geometric invariant theory, I

Studying objects up to a notion of equivalence is an integral concept in any area of mathematics. The appearance of quotients is thus inevitable. In this lecture we begin studying quotients which are particularly well-behaved in the algebro-geometric setting.

3.1. QUOTIENTS IN GEOMETRY AND TOPOLOGY

We will review some classic results about group actions on topological spaces and smooth manifolds.

If G is a group acting on a topological space M then we can consider the set-theoretic quotient M/G which is, by definition, the set of G-orbits. This set is equipped with a natural topology for which the map $M \to M/G$ is continuous. But, in general, this topological space may not even be Hausdorff. To get a nicer behaved quotient we look at a more refined situation.

Suppose that G is a real Lie group acting on a smooth manifold M. We call this action ***proper*** if the map

$$\begin{aligned} G \times M &\to M \times M \\ (g, m) &\mapsto (m, g \cdot m) \end{aligned}$$

is proper; meaning the preimage of any compact set is compact. Equivalently, this is the condition that for any compact sets $K, K' \subset M$ that the subset $\{g \in G \mid gK \cap K'\} \subset G$ is compact.

Denote the stabilizer of a point $x \in M$ by

$$G_x \stackrel{\text{def}}{=} \{g \in G \mid g \cdot x = x\} \subset G. \tag{1}$$

Also, denote the orbit of a point $x \in M$ by

$$\mathbb{O}_x \stackrel{\text{def}}{=} \{y \in M \mid y = g \cdot x\} \subset M. \tag{2}$$

We have the following easy observations:

- if G is compact this condition is automatically true.
- any G-orbit of a proper action is a closed subset of M.

A fundamental result pertaining to quotients spaces in geometry and topology is the following so-called "slice theorem".

THEOREM 3.1.1. *Suppose G acts properly on a smooth manifold M. For every $x \in M$ there exists a locally closed submanifold $S_x \subset M$ (called a* **slice***) containing x which is* *invariant under the action of G_x. Furthermore, there exists an open neighborhood $U \supset \mathbb{O}_x$ such that the natural map*

$$G \times_{G_x} S \xrightarrow{\simeq} U \tag{3}$$

is a homeomorphism.

In particular, $\mathbb{O}_x$ is a smooth closed submanifold of M.

As a corollary one obtains the following.

THEOREM 3.1.2. *Suppose that G acts freely and properly on a smooth manifold M. Then the set of orbits M/G has the structure of smooth manifold with the property that the quotient map*

$$M \to M/G \tag{4}$$

is a smooth principal G-bundle.

So far these theorems take place in the smooth or topological setting. We now move towards quotients in algebraic geometry.

3.2. ALGEBRAIC GROUPS

One has the following hierarchy

$$\begin{aligned}\{\text{groups}\} \supset \{\text{topological groups}\} \supset \{\text{real Lie groups}\} \\ \supset \{\text{complex Lie groups}\} \supset \{\text{algebraic groups}\}\end{aligned} \tag{5}$$

and for actions $G \times X \to X$ we have

$$\begin{aligned}\{\text{action}\} \supset \{\text{continuous action}\} \supset \{\text{smooth action}\} \\ \supset \{\text{holomorphic action}\} \supset \{\text{algebraic action}\}.\end{aligned} \tag{6}$$

An important structure in any of these contexts is *reducibility*. Suppose that we are in a linear situation. A representation (or a linear action) of a group G on a vector space V is called *completely reducible* if it is a direct sum of irreducible representations.

If G is a compact topological group acting linearly on a (real or complex) vector space V then the action is completely reducible. Thus, compactness is enough to guarantee complete irreducibility. But, sometimes we do not want to assume compactness.

Definition 3.2.1. A complex Lie group G is said to be ***reductive*** if

- G has finitely many connected components.
- G contains a compact real Lie subgroup K such that $\mathbf{C} \otimes_{\mathbf{R}} T_eK \simeq T_eG$.

The compact group K is called the reductive form of G.

Example 3.2.2. A basic example of a reductive group is the group $GL(n, \mathbf{C})$ of invertible complex-valued $n \times n$ matrices. In this case the real subgroup $U(n)$ can be taken to be the group of unitary $n \times n$ matrices.

THEOREM 3.2.3. *A reductive linear group action on a complex vector space is completely reducible.*

Let's turn to the nice algebraic context that we will work in for the time being.

Definition 3.2.4. A ***linear algebra group*** G is a subgroup of $GL(V)$, where V is some complex vector space, which is cut out by a finite collection of polynomials. That is, there exists $p_1, \ldots, p_k \in \mathbf{C}[\mathrm{End}(V)]$ such that

$$G = \{g \in GL(V) \mid p_i(g) = 0, \text{for all } i\}. \tag{7}$$

Example 3.2.5. The groups $GL(n, \mathbf{C})$, $SL(n, \mathbf{C})$, $O(n, \mathbf{C})$, $SO(n, \mathbf{C})$, $Sp(n, \mathbf{C})$ are all linear algebra groups. The group $U(n)$ is not a complex linear algebraic group, though it is a real algebraic group.

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Definition 3.2.6. An ***affine algebraic group*** G is an affine algebraic variety together with structure maps $\mu\colon G \times G \to G$, $(-)^{-1}\colon G \to G$, and an element $e \in G$ such that the usual group axioms hold.

An ***algebraic action*** of an algebraic group on an affine algebraic variety X is a map of algebraic varieties $G \times X \to X$ satisfying the axioms of an action.

Proposition 3.2.7. *Any affine algebraic group is isomorphic (as an affine algebraic group) to a linear algebraic group.*

We say an algebraic group G is reductive if it is reductive as a complex Lie group (in the sense above). Here is the main theorem about reductive group actions.

THEOREM 3.2.8. *Let G be a reductive algebraic group acting on an affine variety X. Then*

- $\mathbf{C}[X]^G$ *is a finitely generated* $\mathbf{C}$*-algebra.*
- *If $W, Z \subset X$ are closed, G-invariant, and disjoint then there exists a G-invariant polynomial function $p \in \mathbf{C}[X]$ such that $p|_W \equiv 0$ and $p|_Z \equiv 1$.*

3.3. QUOTIENTS IN AFFINE ALGEBRAIC GEOMETRY

For the next few lectures we will be working in the context where G is an algebraic group acting on an affine algebraic variety X (all defined over $\mathbf{C}$). We will consider X as a topological space using the Zariski topology.

Some key results of invariant theory which we will not prove include the following.

- Every G-orbit in X is a nonsingular algebraic variety (which is not necessarily closed).
- For every orbit $\mathbb{O}$, we can consider the closure $\overline{\mathbb{O}}$. The boundary $\overline{\mathbb{O}} - \mathbb{O}$ is a union of lower dimensional orbits.

In the last lecture we introduced the so-called *geometric invariant theory*, or simply *GIT*, quotient

$$X \,/\!/\, G \overset{\text{def}}{=} \operatorname{Spec}\left(\mathbf{C}[X]^G\right). \tag{8}$$

By the first part of theorem 2.2.8 this is an affine algebraic variety. Also, there is a relationship between the set-theoretic quotient M/G and the GIT quotient. Indeed, given any orbit $\mathbb{O} \in X/G$ we can define the maximal ideal

$$J_{\mathbb{O}} \overset{\text{def}}{=} \{f \in \mathbf{C}[X] \mid f|_{\mathbb{O}} = 0\}. \tag{9}$$

The assignment $\mathbb{O} \mapsto J_{\mathbb{O}}$ defines a continuous map

$$p\colon X/G \to X \,/\!/\, G, \tag{10}$$

where $X \,/\!/\, G$ is equipped with the Zariski topology.

THEOREM 3.3.1. *For X affine and reductive this map is surjective. Moreover two orbits $\mathbb{O}, \mathbb{O}'$ determine the same point in $X \,/\!/\, G$ if and only if the closures of the orbits are disjoint $\overline{\mathbb{O}} \cap \overline{\mathbb{O}}' \neq \emptyset$.*

A related, and even more explicit description of the GIT quotient is granted by the following.

THEOREM 3.3.2. *There is a homeomorphism of topological spaces*

$$X \,/\!/\, G \simeq \{\textit{closed orbits in } M\} \tag{11}$$

which sends $[x] \mapsto$ the unique closed orbit contained in $\overline{\mathbb{O}}_x$.

PROOF. It suffices to show that the closure of any orbit contains a unique closed orbit. For the existence of a closed orbit, recall that boundary of an orbit $\overline{\mathbb{O}} - \mathbb{O}$ is a union of orbits of lower dimension. For uniqueness one relies on Theorem 2.2.8. ☐

As a corollary of this we see that if all orbits are closed then the GIT quotient agrees with the set-theoretic one. Here is a simple example where there is a non-closed orbit.

Example 3.3.3. Suppose that $X = \mathbf{A}^2$ and consider the scaling action of $G = \mathbf{C}^\times$

$$\lambda \cdot (z_1, z_2) = (\lambda z_1, \lambda z_2). \tag{12}$$

Then as a topological space one has

$$\mathbf{A}^2/\mathbf{C}^\times \simeq \mathbb{P}^1 \cup \{0\}. \tag{13}$$

(This is only as a set, $\mathbf{A}^2/\mathbf{C}^\times$ is not a manifold.) On the other hand, the only closed orbit is $\{0\} \subset \mathbf{A}^2$. Indeed, if $0 \neq x \in \mathbf{A}^2$ then $\mathbb{O}_x = \ell_x - \{0\}$ where ℓ_x is the line through x and 0. Thus

$$\mathbf{A}^2 \,/\!/\, \mathbf{C}^\times \simeq \{0\}. \tag{14}$$

Notice that this is consistent with the fact $\mathbf{C}[z_1, z_2]^{\mathbf{C}^\times} = \mathbf{C}$.

In the topological world, we obtain the best structure on the quotient when the G-action is free. It is in this case that the quotient is a smooth manifold (if we start with a smooth manifold) and the quotient map exhibits the original space as a principal bundle. There is an analog of this result in the algebro-geometric world.

To formulate we need to have an algebro-geometric version of a smooth principal bundle. If $x \in X$ is a point in an algebraic variety then let $\widehat{\mathcal{O}_x}$ be the completed local ring at $x \in X$. If $x \in X$ is a non-singular point then this is just the ring of formal power series near x.

Example 3.3.4. Suppose that $X = \operatorname{Spec}(\mathbf{C}[z_1, \ldots, z_n]/(f))$ where f is some polynomial. Then $\widehat{\mathcal{O}_0}$ is isomorphic to $\mathbf{C}[[z_1, \ldots, z_n]]/(f)$.

A morphism of algebraic varieties $f\colon X \to Y$ is called *étale* if for every $x \in X$ the pullback map

$$f^*\colon \widehat{\mathcal{O}_{f(x)}} \to \widehat{\mathcal{O}_x} \tag{15}$$

is an isomorphism.

For us, the analog of a principal G-bundle in algebraic geometry will be an étale G-bundle. A G-equivariant map $f\colon X \to Y$ is an étale G-bundle if for every $y \in Y$ there exists an open neighborhood $U \subset Y$ of y such that $X|_U \to U$ is étale equivalent to the trivial G-bundle $U \times G \to U$.

THEOREM 3.3.5 (Luna Slice). *Suppose that X is a non-singular affine variety equipped with a free G-action where G is a reductive algebraic group. Then $X \mathbin{/\!/} G = X/G$ is a non-singular variety and $X \to X \mathbin{/\!/} G$ is an étale G-bundle.*

3.4. A FUNDAMENTAL EXAMPLE

We end this lecture with a very important example. Let

$$X = \operatorname{End}(\mathbf{C}^n) = \mathbf{A}^{n^2} \tag{16}$$

be the affine space of $n \times n$ matrices. Consider the adjoint action of $G = GL(n, \mathbf{C})$ on X.

Given a matrix $x \in X$ consider its characteristic polynomial

$$\det(x + t\mathbb{1}) = p_n(x) + t p_{n-1}(x) + \cdots + t^{n-1} p_1(x) + t^n \tag{17}$$

where t is some indeterminate.

THEOREM 3.4.1. *The polynomials $p_1, \ldots, p_n \in \mathbf{C}[X]$ are algebraically independent, $GL(n, \mathbf{C})$-invariant, and generate $\mathbf{C}[\operatorname{End}(\mathbf{C}^n)]^{GL(n,\mathbf{C})}$. In particular*

$$\operatorname{End}(\mathbf{C}^n) \mathbin{/\!/} GL(n, \mathbf{C}) \simeq \mathbf{A}^n. \tag{18}$$

The proof of this theorem relies on the following fundamental lemma.

Lemma 3.4.2. *Suppose that an algebraic group G acts on a variety X and suppose that $p_1, \ldots, p_n \in \mathbf{C}[X]$ are G-invariant polynomials. Further, suppose that $H \subset G$ is a subgroup which leaves invariant a subvariety $U \subset X$ with the properties:*

(1) *the polynomials $p_i|_U$ generate the ring $\mathbf{C}[U]^H$;*

(2) *the set $G \cdot U$ is dense in X.*

Then the polynomials $p_1, \dots, p_n$ generate the ring $\mathbf{C}[X]^G$.

PROOF. Let p be a polynomial which is G-invariant. Then, by assumption,

$$p|_U = F(p_1|_U, \dots, p_n|_U) \tag{19}$$

for some polynomial F. Define

$$q \overset{\text{def}}{=} p - F(p_1, \dots, p_n) \in \mathbf{C}[X]. \tag{20}$$

By construction $q|_U \equiv 0$. Moreover, q is G-invariant so that $q|_{G \cdot U} \equiv 0$. By the second assumption we know that $q \equiv 0$. □

Let's turn to the proof of theorem 2.4.1

PROOF OF THEOREM 2.4.1. Fix a basis $\{e_i\}$ of $\mathbf{C}^n$. We apply the lemma to the case where $U \subset X = \mathrm{End}(\mathbf{C}^n)$ is the subspace of diagonal matrices and $H = S_n$ is the permutation group which permutes the $e_1, \dots, e_n$. In particular, a permutation $\sigma \in S_n$ acts on U by the rule

$$\sigma \cdot \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} = \begin{pmatrix} \lambda_{\sigma(1)} & & \\ & \ddots & \\ & & \lambda_{\sigma(n)} \end{pmatrix} \tag{21}$$

We will first show that $\overline{G \cdot U} = X$. Consider the continuous map $\pi \colon X \to \mathbf{C}^n$ which sends a matrix x to the coefficients $p_1(x), \dots, p_n(x)$ of its characteristic polynomial. This map can be shown to be surjective. The subset $V \subset \mathbf{C}^n$ consisting of the coefficients of monic polynomials with distinct roots is an open set. Thus, $\pi^{-1}(V) \subset X$ is an open set which consists of matrices with distinct eigenvalues. Finally, we note that any matrix which has distinct eigenvalues can be diagonalized.

Next, we must show that $p_1|_U, \dots, p_n|_U$ generate $\mathbf{C}[U]^{S_n}$. Let $x \in X$ be the matrix $\mathrm{diag}(\lambda_1, \dots, \lambda_n)$. Then the characteristic polynomial of x is

$$\det(x + t\mathbb{1}) = \sigma_n(\lambda) + \cdots + \sigma_1(\lambda) t^{n-1} + t^n \tag{22}$$

where σ_i is the ith symmetric polynomial. We have already recalled in the last lecture that the elementary symmetric polynomials generate the ring of all symmetric functions. This completes the proof. □