LECTURE 4

# An explicit description of the Hilbert scheme

Today we will prove an explicit characterization of the Hilbert scheme. Let

$$H_n \subset \mathrm{Hom}(\mathbf{C}^n, \mathbf{C}^n)^{\oplus 2} \oplus \mathbf{C}^n \oplus (\mathbf{C}^n)^* \tag{1}$$

is the set of tuples $(X, Y, i, j)$ which satisfy

$$[X, Y] - ij = 0. \tag{2}$$

Also define

$$H_n^s \subset H_n \tag{3}$$

to be the subspace where

- the vector $i(1)$ generates $\mathbf{C}^n$ under the action by $X, Y$. This means that given $v \in \mathbf{C}^n$ there exists integers $k, l \geq 0$ such that $v = X^k Y^l i(1)$.

This last condition is called a ***stability*** condition. We will see many versions of it in future lectures.

THEOREM 4.0.1. *For any $n$ the Hilbert scheme $\mathrm{Hilb}_n(\mathbf{A}^2)$ is a nonsingular algebraic variety of dimension $2n$. Moreover, there is an isomorphism of algebraic varieties*

$$\mathrm{Hilb}_n(\mathbf{A}^2) \simeq H_n^s \,/\!/\, GL(n, \mathbf{C}). \tag{4}$$

There is a similar description of the symmetric product of $\mathbf{A}^2$.

THEOREM 4.0.2. *There is an isomorphism of algebraic varieties*

$$S^n\mathbf{A}^2 \simeq H_n \,/\!/\, GL(n, \mathbf{C}). \tag{5}$$

*Moreover, the natural map $H_n^s \hookrightarrow H_n$ induces the Hilbert–Chow morphism*

$$\pi_{HC} \colon \mathrm{Hilb}_n(\mathbf{A}^2) \to S^n\mathbf{A}^2. \tag{6}$$

## 4.1. A DESCRIPTION OF THE SYMMETRIC PRODUCT

We will begin with the description of the symmetric product which means we will momentarily forget about the stability condition. First, let's carefully describe how $GL(n, \mathbf{C})$ acts on $H_n$. The action is a restriction of the most natural one where $GL(n; \mathbf{C})$ acts on endomorphisms of $\mathbf{C}^n$ by conjugation and acts on $\mathbf{C}^n$ (respectively $(\mathbf{C}^n)^*$) in the defining (respectively antidefining) way. Explicitly, for $g$ an invertible $n \times n$ matrix and $(X, Y, i) \in H'_n$ the action is

$$g \cdot (X, Y, i, j) \stackrel{\text{def}}{=} \left(gXg^{-1}, gYg^{-1}, gi, jg^{-1}\right). \tag{7}$$

The following lemma is a direct calculation and can be found in [Nak99, §2].

**Lemma 4.1.1.** *Suppose $[X,Y]+ij=0$ as above. Let $S\subset \mathbf{C}^n$ be the subset*

$$\sum (X^{n_1}Y^{m_1}\cdots X^{n_k}Y^{m_k})i(\mathbf{C}). \tag{8}$$

*Then $j|_S\equiv 0$.*

Suppose that

$$\mathbb{O}\overset{\text{def}}{=}GL(n,\mathbf{C})\cdot(X,Y,i,j) \tag{9}$$

is a closed orbit. Using $S\subset\mathbf{C}^n$ as in the lemma, we can decompose

$$\mathbf{C}^n=S\oplus S^{\perp}. \tag{10}$$

With respect to this decomposition the matrices $(X,Y,i,j)$ have the form

$$X,Y=\begin{pmatrix}\star & \star\\ 0 & \star\end{pmatrix},\quad i=\begin{pmatrix}\star\\ 0\end{pmatrix},\quad j=\begin{pmatrix}0 & \star\end{pmatrix}. \tag{11}$$

By closedness we can further assume that

$$X,Y=\begin{pmatrix}\star & 0\\ 0 & \star\end{pmatrix},\quad i=\begin{pmatrix}0\\ 0\end{pmatrix},\quad j=\begin{pmatrix}0 & 0\end{pmatrix}. \tag{12}$$

Thus, the condition $[X,Y]+ij=0$ simply becomes $[X,Y]=0$. Choose a basis so that $X,Y$ are both upper triangular. Then, by the closedness assumption we can assume that $X,Y$ are diagonalizable. The equivalence with $S^n\mathbf{A}^2$ associates a closed orbit to the simultaneous eigenvalues of the matrices $X,Y$.

**Remark 4.1.2.** The above argument can be modified to give a short proof that $\mathrm{End}(\mathbf{C}^n)\,/\!/\,GL(n,\mathbf{C})\simeq\mathbf{A}^n$. Indeed, every matrix admits a basis for which it is upper triangular. If we assume a matrix lies in a closed orbit, then we can also assume it is diagonalizable. The equivalence then sends a closed $GL(n,\mathbf{C})$-orbit to its $n$-tuple of eigenvalues.

## 4.2. Relating closed points

Before turning to the proof of theorem 4.0.1, we will give a heuristic argument for the result. First, it turns out that we can simplify the description of $H_n$.

**Lemma 4.2.1.** *Suppose that $(X,Y,i,j)\in H_n^s$. Then $j=0$.*

Let $\widetilde{H}_n^s$ be the subspace consisting of $(X,Y,i)$ with the property that $[X,Y]=0$ and $i$ generates $\mathbf{C}^n$ under the action by $X,Y$. Then as a corollary of this lemma we have $GL(n;\mathbf{C})$-equivariant isomorphism

$$\widetilde{H}_n^s\simeq H_n^s. \tag{13}$$

Let's see how the data of a triple $(X,Y,i)\in\widetilde{H}_n^s$ gives rise to a closed point in $\mathrm{Hilb}_n(\mathbf{A}^2)$. Notice that a closed point in $\mathrm{Hilb}_n(\mathbf{A}^2)$ is, by definition, an ideal $I$ in $\mathbf{C}[z_1,z_2]$ such that $\mathbf{C}[z_1,z_2]/I$ is an $n$-dimensional vector space. Define the linear map

$$\phi_{(X,Y,i)}\colon\mathbf{C}[z_1,z_2]\to\mathbf{C}^n \tag{14}$$

by the formula $\phi_{(X,Y,i)}(f) = f(X,Y)i(1)$. Since $\operatorname{im}\phi$ is invariant under the action of $X, Y$ and contains $\operatorname{im} i$ we see that $\phi$ is surjective by the stability condition. Thus $I = \ker\phi$ is an ideal in $\mathbf{C}[z_1, z_2]$ and $\dim_{\mathbf{C}}(\mathbf{C}[z_1, z_2]/I) = n$.

Next, suppose $I$ is an ideal of codimension $n$ and let $V = \mathbf{C}[z_1, z_2]/I$. Then we have operators $X = z_1$, $Y = z_2$ and $i\colon \mathbf{C} \to V$ defined by $i(1) = 1 \mod I$. It is automatic that $[X, Y] = 0$ and that the stability condition holds.

It is not hard to check that these two operations are mutually inverse to one another, which thus gives an isomorphism of *sets* between $\widetilde{H}_n^s/GL(n, \mathbf{C})$ and codimension $n$ ideals in $\mathbf{C}[z_1, z_2]$.

## 4.3. Proof of the theorem

By the isomorphism (13), we see that theorem 4.0.1 follows from the following result.

**Proposition 4.3.1.** *The Hilbert scheme of n-points on affine space* $\mathrm{Hilb}_n(\mathbf{A}^2)$ *is isomorphic to the nonsingular algebraic variety*

$$\widetilde{H}_n^s \,/\!/\, GL(n, \mathbf{C}). \tag{15}$$

Proof. We will use the algebraic slice theorem as formulated in the previous lecture to argue why $\widetilde{H}_n^s \,/\!/\, GL(n; \mathbf{C})$ is nonsingular. Before taking the quotient, we need to see that $H_n^s$ is non-singular. Consider the map

$$F\colon \mathrm{End}(\mathbf{C}^n)^{\otimes 2} \otimes \mathbf{C}^n \to \mathrm{End}(\mathbf{C}^2) \tag{16}$$

defined by $F(X, Y, i) = [X, Y]$. Let $S$ be the subset of $\mathrm{End}(\mathbf{C}^n)^{\otimes 2} \otimes \mathbf{C}^n$ consisting of triples $(X, Y, i)$ satisfying the stability condition. Observe that $\widetilde{H}_n^s = (F|_S)^{-1}(0)$. To show that $\widetilde{H}_n^s$ is non-singular it suffices to show that the derivative of $F|_S$ has constant rank.

**Lemma 4.3.2.** *Let* $D = D_{(X,Y,i)}(F|_S)$ *be the derivative of the map* $F|_S$ *at* $(X, Y, i) \in S$. *Then*

$$\operatorname{coker} D = \{A \in \mathrm{End}(\mathbf{C}^n) \mid [X, A] = [Y, A] = 0\}. \tag{17}$$

Using this description we can define a map $\operatorname{coker} D \to \mathbf{C}^n$ by the rule $A \mapsto A(i(1))$. Conversely, define a map $\mathbf{C}^n \to \operatorname{coker} D$ by sending $v$ to the endomorphism $A$ which satisfies

$$A_v(X^k Y^l i(1)) = X^k Y^l v \tag{18}$$

for integers $k, l \geq 0$. This is enough to define $A_v$ by the stability condition. These maps are clearly mutual inverses so that $\operatorname{coker} D \simeq \mathbf{C}^n$. Thus, by the constant rank level set theorem (see [Lee13][Theorem 5.12]) we see that $\widetilde{H}_n^s$ is non-singular.

Next, we will apply Luna's slice theorem to see that $\widetilde{H}_n^s \,/\!/\, GL(n, \mathbf{C})$ is non-singular. For this we need to check that the action is free. Suppose that $g \in GL(n, \mathbf{C})$ stabilizes $(X, Y, i)$. This means that

$$gXg^{-1} = X, \quad gYg^{-1} = Y, \quad gi = i. \tag{19}$$

The last equality implies that $\ker(g - \mathbb{1}) \subset \mathbf{C}^n$ contains $\operatorname{im} i$. But the first two equations imply that this subspace is stabilized by $X, Y$. Thus by the stability condition $g = \mathbb{1}$. By the slice theorem

$$Y \overset{\text{def}}{=} \widetilde{H}_n^s /\!/ GL(n, \mathbf{C}) = \widetilde{H}_n^s / GL(n, \mathbf{C}) \tag{20}$$

is a non-singular variety and the map

$$\widetilde{H}_n^s \to Y \tag{21}$$

is an étale principal $GL(n, \mathbf{C})$-bundle.

Now, to characterize $Y$ as the Hilbert scheme of $n$-points on $\mathbf{A}^2$ we need to construct a universal family over $Y$ of 0-dimensional subschemes of size $n$. There is a family on $\mathcal{Y} \to Y$ defined by the natural surjection

$$f(z_1, z_2) \in \mathbf{C}[z_1, z_2] \mapsto f(X, Y)i(1) \in \mathbf{C}^n. \tag{22}$$

To see that it is a universal family we need to show that if $\pi\colon Z \to U$ is any flat family of 0-dimensional closed subschemes of $\mathbf{A}^2$ of size $n$, then there exists a unique morphism $\phi\colon U \to Y$ fitting into the pullback square

$$\begin{array}{ccc} Z & \longrightarrow & \mathcal{Y} \\ {\scriptstyle \pi}\downarrow & & \downarrow \\ U & \xrightarrow[\phi]{} & Y. \end{array} \tag{23}$$

By assumption $\pi_* \mathcal{O}_Z$ is a locally free sheaf of rank $n$ on $U$. Just as in the previous section, define $X, Y$ as the $\mathcal{O}_U$-linear operators acting on $\pi_* \mathcal{O}_Z$ given by multiplication by the coordinate functions $z_1, z_2$ respectively. Also let $i$ be the image of the constant polynomial 1 thought of as an sheaf homomorphism $\mathcal{O}_U \to \pi_* \mathcal{O}_Z$. If we fix an open cover $U = \cup_\alpha U_\alpha$ so that $\pi_* \mathcal{O}_Z$ is trivializable over $U_\alpha$ then we obtain morphisms $U_\alpha \to \widetilde{H}_n^s$ for each $\alpha$. Composing with $\widetilde{H}_n^s \to Y$ these glue together to define a morphism $\phi\colon U \to Y$. By construction we have $\phi^* \widetilde{Y} = Z$. □

**Remark 4.3.3.** Notice that the proof of this result didn't rely much of our knowledge of the GIT quotient. Since the $GL(n, \mathbf{C})$ action on $\widetilde{H}_n^s$ is free, the GIT quotient is the same as the set-theoretic quotient.

# Bibliography


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