LECTURE 1

The Heisenberg algebra

Last time we finished the computation of the generating function for the Poincaré polynomials of the Hilbert scheme of points on $\mathbf{C}^2$. We saw that

$$\sum_{n\geq 0} P_t(\mathrm{Hilb}_n(\mathbf{C}^2)) = \prod_{m\geq 1} \frac{1}{1-t^{2m-2}q^m}. \tag{1}$$

Notice that if we specialize $t = 1$ we uncover the infinite product $\prod_{m\geq 1}(1-q^m)^{-1}$ (which is also the generating function for partitions).

Consider the polynomial algebra on an infinite number of generators

$$\mathbf{C}[x_1, x_2, \ldots]. \tag{2}$$

Of course, this is an infinite-dimensional space so it does not make sense to contemplate its dimension. However, if we introduce the operator

$$L_0 \stackrel{\text{def}}{=} \sum_m m x_m \frac{\partial}{\partial x_m} \tag{3}$$

then the individual L_0-eigenspaces are finite dimensional. Indeed, let

$$F^{(l)} \stackrel{\text{def}}{=} \{v(x_1, x_2, \ldots) \mid L_0 v = lv\}. \tag{4}$$

Notice that $F^{(l)} = 0$ for $l < 0$, $F^{(0)} = F^{(1)} = \mathbf{C}$ and $F^{(2)} = \mathrm{span}\{x_1^2, x_2\} = \mathbf{C}^2$. By definition the graded dimension (or q-dimension) of this vector space with respect to the grading determined by L_0 is

$$\sum_{l\geq 0} q^l \dim F^{(l)}. \tag{5}$$

An easy exercise identifies this graded dimension with the same infinite product $\prod_{m\geq 1}(1-q^n)^{-1}$. What is the connection between these two computations?

1.1. The (super) Heisenberg algebra

As a vector space, the Heisenberg Lie algebra $\mathfrak{h}$ is $\mathbf{C}((t)) \oplus \mathbf{C} \cdot K$. The vector K is central, and the Lie bracket is

$$[f(t), g(t)] = \mathrm{Res}(f \mathrm{d}g) \cdot K \tag{6}$$

where Res is the residue at $t = 0$. Explicitly, on monomials this bracket reads

$$[t^n, t^m] = m\delta_{n+m,0}K. \tag{7}$$

We will denote the basis of monomials by $b_n \stackrel{\text{def}}{=} t^n$.

For any $\kappa \in \mathbf{C}^\times$ the Lie algebra $\mathfrak{h}$ acts on

$$\mathbf{C}[x_1, x_2, \ldots] \tag{8}$$

by the rules that b_0 acts trivially,

$$m > 0, \qquad b_m \mapsto \kappa m \frac{\partial}{\partial x_m} \tag{9}$$

and

$$m < 0, \qquad b_m \mapsto x_{-m}. \tag{10}$$

Since b_0 acts trivially this representation is not irreducible. But if we remove b_0 to form the Lie algebra $\mathfrak{h}' = \mathfrak{h}/\mathbf{C}b_0$ then this is an irreducible $\mathfrak{h}'$-representation which we denote by F_κ—we call this the Heisenberg Fock module. Notice that as a vector space we can identify this representation with

$$F_\kappa = \mathrm{Sym}(t^{-1}\mathbf{C}[t^{-1}]). \tag{11}$$

We can extend the action of $\mathfrak{h}'$ on F_κ by the element $L_0 \stackrel{\text{def}}{=} \sum_m m x_m \frac{\partial}{\partial x_m}$. Then we have the character formula

$$\mathrm{tr}_{F_\kappa}(q^{L_0}) = \sum_{l \geq 0} q^l \dim F_\kappa^{(l)} = \prod_{m \geq 1} \frac{1}{1-q^m} \tag{12}$$

which agrees with the graded dimension from the introduction.

More generally, suppose that V is any vector space equipped with a non-degenerate bilinear form $\langle -, - \rangle$. The Heisenberg Lie algebra associated to V is the Lie algebra $\mathfrak{h}(V)$ whose underlying vector space is

$$\mathfrak{h}(V) \stackrel{\text{def}}{=} V \otimes_{\mathbf{C}} \mathbf{C}((t)) \oplus \mathbf{C} \cdot K \tag{13}$$

and the bracket is

$$[v \otimes f(t), w \otimes g(t) = \langle v, w \rangle \mathrm{Res}(f\mathrm{d}g)K \tag{14}$$

Let $\mathfrak{h}'(V)$ be the quotient Lie algebra by the central subalgebra $V \otimes 1$. The Fock module associated to this Heisenberg algebra is, as a vector space, given by

$$F_k(V) = \mathrm{Sym}\left(V \otimes t^{-1}\mathbf{C}[t^{-1}]\right). \tag{15}$$

This admits an irreducible action of $\mathfrak{h}'(V)$ by the same formulas as above.

The assignment $V \mapsto \mathfrak{h}'(V)$ defines a functor from the category of vector spaces with bilinear forms to the category of Lie algebras. In fact, the same formulas define a functor from the category of super vector spaces equipped with super bilinear forms to the category of super Lie algebras.

Recall that a super vector space is a splitting of a vector space into even and odd components

$$V = V_{even} \oplus \Pi V_{odd}. \tag{16}$$

We write $|v| = 0$ if $v \in V_{even}$ and $|v| = 1$ if $v \in V_{odd}$. A super bilinear form on V is a bilinear functional satisfying

$$\langle v, w \rangle = (-1)^{|v||w|} \langle w, v \rangle. \tag{17}$$

For a super vector space V we will call $\mathfrak{h}'(V)$ the (reduced) Heisenberg super Lie algebra associated to V. When V is purely odd this is sometimes called the infinite-dimensional Clifford algebra associated to V.

The construction of the Fock module $F_\kappa(V)$ is almost identical to the purely bosonic case. The subtlety is that one should interpret the symmetric algebra as the *graded* symmetric algebra. The symmetric algebra $V \mapsto S^\bullet(V)$ is a functor from vector spaces to commutative algebras. The graded symmetric algebra $V \mapsto \mathrm{Sym}(V)$ is a functor from super vector spaces to super commutative algebras. If $V = V_{even} \oplus \Pi V_{odd}$ then as an ordinary vector space one has

$$\mathrm{Sym}(V) = S(V_{even}) \oplus \wedge(V_{odd}) \tag{18}$$

where $\wedge(-)$ is the exterior algebra.

Thus, as a vector space one can identify the general Fock module associated to a super vector space $V = V_{even} \oplus \Pi V_{odd}$ as

$$F_\kappa(V) = S\left(V_{even} \otimes t^{-1}\mathbf{C}[t^{-1}]\right) \otimes \wedge\left(V_{odd} \otimes t^{-1}\mathbf{C}[t^{-1}]\right). \tag{19}$$

The operator L_0 acting on $F_\kappa(V)$ is defined as above. One can show that

$$\mathrm{tr}_{F_\kappa(V)}\, q^{L_0} = \prod_{m \geq 1} \frac{(1+q^m)^{\dim V_{odd}}}{(1-q^m)^{\dim V_{even}}}. \tag{20}$$

If we take the super trace as opposed to the ordinary trace then this becomes

$$\mathrm{str}_{F_\kappa(V)}\, q^{L_0} = \prod_{m \geq 1} (1-q^m)^{\dim V_{odd} - \dim V_{ev}} \tag{21}$$

1.2. Borel–Moore homology

The goal for the next few lectures is to construct a geometric action of the Heisenberg algebra on the cohomology of Hilbert schemes.

Let X be any topological space which admits an embedding into $\mathbf{R}^m$ as a closed subspace for some m. The Borel–Moore homology of X is defined to be

$$H_i^{BM}(X) \stackrel{\text{def}}{=} H^{m-i}(\mathbf{R}^m, \mathbf{R}^m - X). \tag{22}$$

This definition is independent of the embedding chosen. If X can be embedded into a closed, oriented m-manifold M as a closed subspace then

$$H_i^{BM}(X) = H^{m-i}(M, M - X). \tag{23}$$

Thus, if X is itself a closed, oriented m-manifold then

$$H_i^{BM}(X) = H^{m-i}(X). \tag{24}$$

For example $H_i^{BM}(\mathbf{C}^n)$ is zero if $i \neq 2n$ and one-dimensional for $i = 2n$. Notice that by Poincaré duality the ordinary homology groups satisfy $H_i(X) = H_c^{m-i}(X)$ where the subscript c denotes cohomology with compact supports. Most of the features of Borel–Moore homology we will use can be directly gleaned from standard facts about relative cohomology. For example, Borel–Moore homology enjoys a Künneth formula

$$H_\bullet^{BM}(X \times Y) \xrightarrow{\simeq} H_\bullet^{BM}(X) \otimes H_\bullet^{BM}(Y). \tag{25}$$

A key feature of Borel–Moore homology is that oriented manifolds, regardless of whether they are compact, have fundamental classes. Indeed, for X a smooth manifold of dimension m the above definition shows that $H_m^{BM}(X) = H^0(X)$. When X

is connected we denote the generator of this by $[X] \in H_m^{BM}(X)$. As another example, there are long exact sequences for computing Borel–Moore homology; we will not need those yet.

Let us dive into some more refined properties of this homology theory. From the definition above one sees that Borel–Moore homology is a covariant functor for proper maps. If $f\colon X \to Y$ is proper then we have a pushforward, or integration, map

$$f_*\colon H_\bullet(X) \to H_\bullet(Y). \tag{26}$$

Notice that this preserves the natural grading on Borel–Moore homology. If $g\colon Y \to Z$ is another proper map then $(g \circ f)_* = g_* \circ f_*$. Borel–Moore homology is contravariant for open embeddings. If $U \subset X$ is an open subset then there is a natural restriction map

$$H_\bullet^{BM}(X) \to H_\bullet^{BM}(U). \tag{27}$$

There is also a sort of product structure on Borel–Moore homology, which bears a connection to an intersection product. Let Z, Z' be two closed subsets of an m-dimensional oriented manifold M. Recall the cup product in cohomology

$$\cup\colon H^{m-i}(X, X-Z) \times H^{m-j}(X, X-Z') \to H^{2m-i-j}(X, X-(Z \cup Z')). \tag{28}$$

By the definition of Borel–Moore homology this gives rise to the *intersection pairing*

$$\cap\colon H_i^{BM}(Z) \times H_j^{BM}(Z') \to H_{i+j-m}^{BM}(Z \cap Z'). \tag{29}$$

While the groups $H_i^{BM}(Z)$, etc. do not depend on M, this intersection product does!