LECTURE 1

Instantons and four-dimensional gauge theory

We turn to an important result behind a construction of [**ADHM**] which provides a description of the (framed) moduli space of instantons on $\mathbf{R}^4$ in terms of finite dimensional matrices akin to the presentation for the moduli space of (framed) torsion-free sheaves on $\mathbf{P}^2$. The history of this subject is long and non-linear, but we refer to Chapter 3 of [**DonaldsonKronheimer**] for a nice overview containing a complete set of references.

An instanton is an anti-self dual (ASD) connection for the group $U(r)$, which on $\mathbf{R}^4$ we must require to be of finite energy meaning $\int_{\mathbf{R}^4} |F|^2 \mathrm{d}^4 x < \infty$. In this case the quantity $n(A) = \frac{1}{8\pi^2}\int_{\mathbf{R}^4} |F|^2 \mathrm{d}^4 x$ is an integer called the instanton number. (Topologically this is the second Chern class of the underlying vector bundle.) The main object of study in the next few lectures is the moduli space of instantons of rank r and instanton number n.

1.1. Aspects of four-dimensional gauge theory

First, let's introduce some relevant objects for studying four-dimensional gauge theory. Most of what we discuss here can (and should) be carried over to the case of a Riemannian four-manifold (M, g); but we will focus on $\mathbf{R}^4$ with its flat metric.

Let $T \cong \mathbf{R}^4$ be the fundamental representation of $SO(4)$. Notice that $\wedge^2 T \simeq \mathfrak{so}(4)$, the adjoint representation of $SO(4)$. Acting on two-forms, the Hodge star operator satisfies $\star^2 = \mathbb{1}$ and determines a decomposition

$$\wedge^2 T \simeq \wedge^+ \otimes \wedge^- \tag{1}$$

where $\star$ acts on $\wedge^\pm$ with eigenvalues ± 1.

Let S be a complex two-dimensional vector space equipped with a Hermitian metric. Also, fix an orientation $\lambda \in \wedge^2 S^*$ with length $|\lambda| = 2$. Then, we can define an anti-linear map $J\colon S \to S$ by

$$\langle x, Jy \rangle = \lambda(x, y). \tag{2}$$

This map satisfies $J^2 = -\mathbb{1}$ and hence endows S with the structure of a quaternionic vector space. Conversely, given J and the hermitian metric we can recover λ. This gives an isomorphism between $SU(2)$ and $Sp(1)$ (=the group of unit quaternions).

Let $S^\pm$ be a pair of such vector spaces and consider the space

$$\mathrm{Hom}_J(S^+, S^-) \subset \mathrm{Hom}(S^+, S^-) \tag{3}$$

of complex linear maps which intertwine the J-actions (so are maps linear over quaternions). This is a four-dimensional real vector space and determines a real slice in the space of all complex linear maps. It carries an induced Euclidean metric so that the unit vectors in $\mathrm{Hom}_J(S^+, S^-)$ are real linear maps $S^+ \to S^-$ which preserve the hermitian metrics and symplectic forms.

Definition 1.1.1. A ***spin structure*** on the four-dimensional Euclidean vector space T is a pair $S^\pm$ as above with an isomorphism

$$\Gamma\colon T \to \mathrm{Hom}_J(S^+, S^-) \tag{4}$$

preserving the Euclidean metrics.

The symmetry group of this data is $Spin(4) = SU(2) \times SU(2)$, the double cover of $SO(4)$. In this local geometric picture, there is a basically a unique spin structure up to permuting $S^\pm$.

Suppose that $e \in T$ and consider $\Gamma(e)\colon S^+ \to S^-$. and its hermitian adjoint $\Gamma(e)^\dagger\colon S^- \to \S^+$. These satisfy $\Gamma(e)^\dagger\Gamma(e) = 1$ if $|e| = 1$ and

$$\Gamma(e)^\dagger\Gamma(e') + \Gamma(e')^\dagger\Gamma(e) = 0 \tag{5}$$

whenever $(e, e') = 0$. From this we obtain an action of $\wedge^2 T \simeq \mathfrak{so}(4)$ on S^+ by the formula

$$(e \wedge e') \cdot s = -\Gamma(e)^\dagger\Gamma(e')s, \tag{6}$$

where $(e, e') = 0$. Using the metric we can extend this to an action of $\wedge^2 T \simeq \mathfrak{so}(4)$. Now it is clear that $\wedge^- \subset \wedge^2 T$ acts trivially and we get an isomorphism

$$\wedge^+ \xrightarrow{\simeq} \mathfrak{su}(S^+). \tag{7}$$

This is an infinitesimal manifestation of the isomorphism $SO(3) \simeq SU(2)$.

In fact, $\wedge^{2,-} T$ acts trivially on S^+ and there is a natural isomorphism

$$\rho\colon\ \wedge^{2,+} T \simeq \mathfrak{su}(2^+). \tag{8}$$

Now, suppose that U is a two-dimensional complex vector space. For any $0 \neq u \in U$ we have an exact sequence

$$0 \to \mathbf{C} \xrightarrow{u} U \xrightarrow{u \wedge -} \wedge^2 U \to 0 \tag{9}$$

given by wedge product with u. Note that a nonzero element $\theta \in \wedge^2 U$ gives an identification $\wedge^2 U \simeq \mathbf{C}$.

Fix a hermitian metric on U (an hence on all of U's wedge powers) such that $|\theta| = 2$. In local coordinates we can think about U as the space of constant coefficient holomorphic one-forms on $\mathbf{C}^2$ and we take $\theta = \mathrm{d}z_1\mathrm{d}z_2$. The hermitian structure allows us to consider the adjoint

$$(u \wedge -)^\dagger\colon \mathbf{C} =_\theta \wedge^2 U \to U. \tag{10}$$

Lemma 1.1.2. *Let $S^+ = \mathbf{C} \oplus \wedge^2 U = \mathbf{C} \oplus \mathbf{C}$ and $S^- = U$. Then, the map*

$$\Gamma \stackrel{\mathrm{def}}{=} u \wedge (-) + (u \wedge -)^\dagger\colon\ \oplus U \xrightarrow{\simeq} \mathrm{Hom}_J(S_+, S_-) \tag{11}$$

is an isomorphism which preserves the Euclidean metrics. In particular, U is equipped with a canonical spin structure.

1.2. MODULI OF INSTANTONS

Let X be a real four-dimensional hyperkähler manifold and suppose $E \to X$ is a Hermitian vector bundle of rank r. Let $\mathcal{A}$ be the space of connections on E which are compatible with the metric. The tangent space at a fixed connection $A \in \mathcal{A}$ is

$$T_A\mathcal{A} \simeq \Omega^1(X, \mathfrak{u}(E)). \tag{12}$$

On $\mathcal{A}$ we have the following metric

$$(\alpha, \beta) \stackrel{\text{def}}{=} -\int_X \mathrm{tr}(\alpha \wedge \star\beta), \quad \alpha, \beta \in \Omega^1(X, \mathfrak{u}(E)). \tag{13}$$

Proposition 1.2.1. *Together with this metric, the hyperkähler structure on X endow $\mathcal{A}$ with the structure of an infinite-dimensional hyperkähler manifold.*

Let $\mathcal{G}$ be the group of gauge transformations associated to the bundle E. Its Lie algebra is $\mathrm{Lie}(\mathcal{G}) = \Gamma(X, \mathfrak{u}(E))$ where $\mathfrak{u}(E) \subset \mathrm{End}(E)$ is the bundle of unitary endomorphisms. Its dual is given by distributional sections of $\wedge^4 \mathrm{T}^*_X \otimes \mathfrak{u}(E)$.

The standard $\mathcal{G}$ action on $\mathcal{A}$ by gauge transformations admits a hyperkähler moment map

$$\mu = (\mu_I, \mu_J, \mu_K) \colon \mathcal{A} \to \mathbf{R}^3 \otimes \Omega^4(X, \mathfrak{u}(E)) \subset \mathbf{R}^3 \otimes \overline{\Gamma}(X, \wedge^4 \mathrm{T}^*_X \otimes \mathfrak{u}(E)). \tag{14}$$

Explicitly

$$\mu_a(A) = F_A \wedge \omega_a, \quad a = I, J, K \tag{15}$$

where $\omega_a \in \Omega^2(X)$, $a = I, J, K$ are the Kähler forms associated with the complex structures I, J, K respectively.

Lemma 1.2.2. *The two-forms $\omega_a, a = I, J, K$ are self-dual two-forms on X. Furthermore, any self-dual two-form can be written as a linear combination of these two-forms.*

From this lemma we see that

$$F_A \wedge \omega_a = 0, \quad a = I, J, K \qquad \Longleftrightarrow \qquad F_A^+ = 0. \tag{16}$$

In other words $\mu^{-1}(0)$ is the space of anti-self-dual connections

$$\mu^{-1}(0)/\mathcal{G} = \{A \mid F_A^+ = 0\}/\mathcal{G} \tag{17}$$

is the moduli space of anti-self-dual connections. Of course, the $\mathcal{G}$ action is very far from being free, so this moduli space has singularities. The thing that is different in this context is that the spaces $\mu^{-1}(0)$ and $\mathcal{G}$ are infinite-dimensional, but the quotient above is finite dimensional.

The constructions above carry over to the non-compact case at the expense of having to slightly modify $\mu^{-1}(0)$. When $X = \mathbf{C}^2$ we quotient out by gauge transformations which converge to the identity at $\infty \in \mathbf{C}^2$. Thus, we should extend our gauge field to the one-point compactification $S^4 = \mathbf{C}^2 \cup \{\infty\}$ and consider the the moduli space

$$\mathcal{M}^{fr}_{ASD}(n, r) \stackrel{\text{def}}{=} \{\text{ASD connections } A \text{ on } E \mid E_\infty \simeq \mathbf{C}^r\}/\simeq. \tag{18}$$

Here we only look at connection of instanton number n.

We head towards a relationship between this moduli space and the sort of reductions that we have been studying [**ADHM**]. Let V, W be Hermitian vector spaces of dimension n, r respectively and consider the vector space

$$H_{n,r} \stackrel{\text{def}}{=} \mathrm{T}^* \operatorname{End}(V) \oplus \mathrm{T}^* \operatorname{Hom}(W, V) \tag{19}$$

which is of dimension $2n^2 + 2nr$. The map $\mu_{\mathbf{R}} \colon H_{n,r} \to \mathfrak{u}(n)$ defined by

$$\mu_{\mathbf{R}}(B_1, B_2, i, j) = \frac{\mathrm{i}}{2}\left([B_1, B_1^\dagger] + [B_2, B_2^\dagger] + ii^\dagger - j^\dagger j\right) \tag{20}$$

is a real moment map for the canonical $U(n)$ action on $H_{n,r}$. The map $\mu_{\mathbf{C}} \colon H_{n,r} \to \mathfrak{gl}(n)$ defined by

$$\mu_{\mathbf{C}}(B_1, B_2, i.j) = [B_1, B_2] + ij. \tag{21}$$

is an algebraic/holomorphic moment map for the canonical $GL(n, \mathbf{C})$ action on $H_{n,r}$. Together these define the hyperkähler moment map

$$\mu = (\mu_1, \mu_2, \mu_3) \stackrel{\text{def}}{=} (\mu_{\mathbf{R}}, \operatorname{Re} \mu_{\mathbf{C}}, \operatorname{Im} \mu_{\mathbf{C}}). \tag{22}$$

Let

$$\mathcal{M}_0(n, r) = \mu^{-1}(0)/U(n) = \mu_{\mathbf{R}}^{-1}(0) \cap \mu^{-1}(0)/U(n) \tag{23}$$

be the hyperkähler reduction of $H_{n,r}$ associated to $\zeta = 0$. This space is singular, but we can look at its regular locus

$$\mathcal{M}_0^{reg}(n, r) = \{[(B_1, B_2, i, j)] \mid \text{stabilizer of } (B_1, B_2, i, j) \text{ is trivial}\} \subset \mathcal{M}_0(n, r). \tag{24}$$

Next time we will spend some time partially explaining the main result.

THEOREM 1.2.3 ([**ADHM**]). *There is a bijective correspondence*

$$\mathcal{M}_{ASD}^{fr}(n, r) \simeq \mathcal{M}_0^{reg}(n, r). \tag{25}$$