LECTURE 1

The cohomology of the Hilbert scheme, II

Today we will finish the computation of the Poincaré polynomial of the Hilbert scheme.

1.1. THE MORSE FUNCTION FOR THE HILBERT SCHEME

For the rest of this lecture we let X_n denote the Hilbert scheme of n points on $\mathbf{C}^2$. In what follows we will use the description of X_n as a Kähler quotient

$$X_n = \mu_{\mathbf{C}}^{-1}(0) \cap \mu_{\mathbf{R}}^{-1}(i\mathrm{d}\chi)/U(n) \tag{1}$$

where $\mu_{\mathbf{C}}$ and $\mu_{\mathbf{R}}$ are the complex and real moment maps. We take χ to be the determinant.

Consider the action of the compact two-torus $T^2 = U(1) \times U(1)$ on $\mathbf{C}^2$ which simply rotates each coordinate

$$(t_1, t_2)\colon (z_1, z_2) \mapsto (t_1 z_1, t_2 z_2). \tag{2}$$

The action is Hamiltonian with moment map

$$\mu_0\colon \mathbf{C}^2 \to (\mathfrak{t}^*)^2 \tag{3}$$

given by $(z_1, z_2) \mapsto (\frac{\mathrm{i}}{2}|z_1|^2, \frac{\mathrm{i}}{2}|z_2|^2)$. This further induces an action of T^2 on the Hilbert scheme defined by

$$(t_1, t_2)\colon [(B_1, B_2, i, 0)] \mapsto [(t_1 B_1, t_2 B_2, i, 0)]. \tag{4}$$

The action is still Hamiltonian with moment map

$$\mu\colon X \to (\mathfrak{t}^*)^2 \tag{5}$$

defined by

$$\mu([(B_1, B_2, i, 0)]) = \left(\frac{\mathrm{i}}{2}\|B_1\|^2, \frac{\mathrm{i}}{2}\|B_2\|^2\right) \tag{6}$$

For ε some real number we consider the element $\xi = -2\mathrm{i}(1, \varepsilon) \in (\mathfrak{t}^2)^*$ and the corresponding function $f\colon X_n \to \mathbf{R}$ defined by

$$f([B_1, B_2, i, 0]) = \langle \mu([B_1, B_2, i, 0]), \xi \rangle = \|B_1\|^2 + \varepsilon \|B_2\|^2. \tag{7}$$

We will assume that ξ is generic which amounts to choosing $0 < \varepsilon \ll 1$. It is in this case that we can guarantee that the fixed point set $X_n^{T^2}$ is in bijective correspondence with the critical points of f.

1.2. Torus fixed points

A point $[B_1, B_2, i, 0]$ is a fixed point in $X_n^{T^2}$ if and only if there exists a homomorphism $\lambda\colon T^2 \to U(V)$ such that

$$t_\alpha B_\alpha = \lambda(t)^{-1} B_\alpha \lambda(t), \quad \alpha = 1, 2$$
$$i = \lambda(t)^{-1} i.$$

Such data provides a decmoposition

$$V = \bigoplus_{k,l} V(k,l) \tag{8}$$

where

$$V(k,l) = \{v \in V \mid \lambda(t) \cdot v = t_1^k t_2^l v\}. \tag{9}$$

Notice that if $v \in V(k,l)$ then $B_1 v \in V(k-1,l)$ since

$$\lambda(t) \cdot (B_1 v) = (t_1^{-1} B_1)(\lambda(t) \cdot v) = t_1^{k-1} t^l v. \tag{10}$$

Similarly $B_2 v \in V(k, l-1)$. Thus we can think about the B_α's as defining a two-dimensional grid of vector spaces whose nodes are labeled by $V(k,l)$ and for which each square commutes since $[B_1, B_2] = 0$ by the moment map condition. We also note that $i(1) \in V(0,0)$ by the fixed point condition above.

Proposition 1.2.1. *(1) The dimension of $V(k,l)$ is at most one. Moreover $V(k,l) = 0$ if either $k > 0$ or $l > 0$.*
(2) $\dim V(k-1,l) \le \dim V(k,l) \ge \dim V(k,l-1)$ if $k, l \le 0$.
(3) If for $k, l \le 0$ the spaces $V(k,l)$ and $V(k-1,l)$ (respectively $V(k,l-1)$) are nonzero then $B_1\colon V(k,l) \to V(k-1,l)$ (respectively $B_2\colon V(k,l) \to V(k,l-1)$) is nonzero.

Proof. All statements follow from the stability condition. Since $V = \mathbf{C}^n$ is spanned by elements of the form $B_1^p B_2^1 i(1)$ we see that $V(k,l) = 0$ if either k or l are positive. Now $B_1\colon V(0,0) \to V(-1,0)$. By stability we know that $i(1)$ must span $V(0,0)$ and that $B_1 i(1)$ must span $V(-1,0)$; thus $\dim V(0,0) \ge \dim V(-1,0)$. By induction we see the second part of assertion (1). The remaining assertions can be proved similarly. ☐

From this proposition we see that the entire data of a fixed point is the data of a Young diagram.

1.3. Young diagrams and a character formula

If $Z \in X_n$ is a fixed point then the tangent space $T_Z X$ is naturally a T^2-representation. In this section we state a formula for the character of this representation as an element of the representation ring $\mathbf{Z}[T_1^{\pm}, T_2^{\pm}]$. Here T_α denotes the one-dimensional representation given by $T_\alpha\colon (t_1, t_2) \mapsto t_\alpha$.

Above we saw that such a fixed point can be identified with a Young diagrm. Let D be a Young diagram and let $s \in D$ be a box. Denote by $r(s)$ (respectively $a(s)$) the number of boxes to the right (respectively above) s.

Proposition 1.3.1. *Let $Z \in X^{T^2}$. The T^2-character of the complex representation $\mathrm{T}_Z X$ is*

$$\sum_{s \in D} \left(T_1^{r(s)+1} T_2^{-a(s)} + T_1^{-r(s)} T_2^{a(s)+1} \right). \tag{11}$$

PROOF. We don't give a full proof, but we mention the key ideas. If $Z = (B_1, B_2, i, 0)$ is any element of the Hilbert scheme then we have provided a model for the tangent space $T_Z X_n$ as the cohomology of a certain three term cochain complex.

The cochain complex is of the form

$$\mathrm{End}(V)[1] \xrightarrow{a} \mathrm{Hom}(V, Q \otimes V) \oplus \mathrm{Hom}(W, V) \oplus \mathrm{Hom}(V, \wedge^2 Q \otimes W) \xrightarrow{b} \mathrm{End}(V)[-1] \tag{12}$$

where $Q = \mathbf{C}^2$. The precise form of the maps will not be important, but we recall that $T_Z X_n = \ker b / \operatorname{im} b$ is the middle cohomology, and that there is no other cohomology. For this complex to be T^2-equivariant one must take Q to transform as the fundamental $U(2)$-representation.

The character of $T_Z X_n$ can then be understood as the graded character of the above complex of T^2-representations. In other words, we view the above complex as a virtual T^2-representation. For example, the character of the virtual representation

$$\mathbf{C}[1] \oplus Q \oplus \wedge^2 Q[-1] \tag{13}$$

is

$$-1 + T_1 + T_2 - T_1 T_2. \tag{14}$$

From the above proposition we also know that V has T^2-character of the form

$$V = \sum_{j=1}^{\nu_1} \sum_{i=1}^{\nu_i'} T_1^{-j+1} T_2^{-i+1} \tag{15}$$

where ν_i, ν_j' denotes the number of boxes in the ith column and jth row. Combining these assertions one can proceed with a computation of the desired character. We refer, as always, to [**NakajimaBook**]. □

From this proposition we can use the perfectness of the Morse function $f \colon X_n \to \mathbf{R}$ to compute the Poincaré polynomial of X_n.

THEOREM 1.3.2. *The Poincaré polynomial of X_n is*

$$P_t(X_n) = \sum_{\nu} t^{2(n - \ell(\nu))} \tag{16}$$

where the sum is over all partitions ν of n and $\ell(\nu)$ is the length of the partition.

PROOF. By perfectness of the Morse function we have

$$P_t(X_n) = \sum_{\nu} t^{d_\nu} \tag{17}$$

where d_ν is the index of the fixed point Z corresponding to the partition ν. From the exact form of the element $\xi = -2\mathrm{i}(1, \varepsilon)$ we see that the index is the sum of the real dimensions of the weight spaces in $T_Z X$ which satisfy

- the weight of t_1 is negative or
- the weight of t_1 is zero and the weight of t_2 is negative.

From the proposition above we see that the second item is impossible. Also by the proposition, we see that the sum of the complex dimensions of the weight spaces where the weight of t_1 is negative is equal to the number of boxes s in the Young diagram with $r(s) > 0$; this is the same as $n - \ell(\nu)$. □

1.4. FINISHING THE PROOF

We can now prove that

THEOREM 1.4.1. *The generating function of the Poincaré polynomial of the Hilbert schemes on* $\mathbf{C}^2$ *is*

$$\sum_{n\geq 0} P_t(X_n)q^n = \prod_{m\geq 1} \frac{1}{1-t^{2m-2}q^m}. \tag{18}$$

PROOF. Let $p(n,\ell)$ be the number of partitions of n of length ℓ. Then it is a standard fact that

$$\sum_{n,\ell} p(n,\ell)x^n y^\ell = \prod_{m\geq 1} \frac{1}{1-yx^m}. \tag{19}$$

From the above theorem we see that

$$\sum_{n\geq 0} P_t(X_n)q^n = \sum_{n,\ell} p(n,\ell)(qt^2)^n t^{-2\ell} \tag{20}$$

and the result follows. □