LECTURE 1

Introduction to the class

Popularized and pioneered by Grothendieck, Hilbert schemes are among the most fundamental moduli spaces in algebraic geometry. Given an algebraic variety one can study the space which parameterizes all possible subschemes of the fixed variety; this space is called a Hilbert scheme. Throughout this course we will study the Hilbert scheme of dimension zero subvarieties (points) of smooth algebraic surfaces. There is a nice balance of richness and accessibility in the context of Hilbert schemes of points in algebraic surfaces. On one hand, the Hilbert scheme of points of algebraic curves agrees with the symmetric product of the curve; so this is a rather trivial case of Hilbert schemes. On the other hand, for smooth algebraic varieties of dimension at least three, the Hilbert scheme of points is generally singular. It is in the case of dimension zero subschemes of a smooth algebraic variety where the Hilbert scheme is smooth and irreducible.

In the case that the algebraic surface is affine space $\mathbf{A}^2$, one think about Hilbert schemes as a particular moduli space of rank one torsion-free sheaves on the projective variety $\mathbb{P}^2$. More generally one can consider moduli spaces of torsion-free sheaves of higher rank. These moduli spaces are amenable to similar tools and techniques that one uses for Hilbert schemes.

Part of the goal of this course is to elucidate algebro-geometric properties of Hilbert schemes of points and moduli spaces of torsion-free sheaves on smooth algebraic surfaces. For the lecturer, however, perhaps most fascinating is the connection between these moduli spaces and at least three other topics:

1. Moduli spaces of instantons on $\mathbf{R}^4$ and more generally singularities of ALE type.
2. Representation theory of infinite-dimensional Lie algebras such as affine Kac–Moody algebras.
3. String theory and M theory. Specifically the infamous theory $\mathfrak{X}$ which one can think about as a twist of the worldvolume theory of fivebranes in M theory.

We will mostly be following the books [Nak99; Kir16].

1.1. Quiver representations and instantons

Solutions to the anti-self-dual Yang–Mills equations on a four-dimensional manifold M are called *instantons*. Amazingly, by work of Atiyah, Drinfeld, Hitchin, and Manin (ADHM) for $M = \mathbf{R}^4$ gauge equivalent classes of such connections can be described in terms of solutions of some quadratic equations for certain finite-dimensional matrices [Ati+78]. This relies on a presentation of $\mathbf{R}^4 = \mathbf{C}^2$ as a hyper Kähler space. Let us briefly sketch this approach for $U(N)$ instantons of 'charge k'. Here, charge is given by $\int_{S^4} F_A \wedge F_A$ normalized so that it is an integer.

Fix the following data:

- a pair of complex vector spaces V, W of dimensions k, N, respectively.
- a pair of complex endomorphisms $X, Y \colon V \to V$.
- a pair of linear maps $i \colon W \to V$ and $j \colon V \to W$.

These are required to satisfy the ADHM equation

$$XY - YX + ij = 0, \tag{1}$$

together with a non-degeneracy (or stability) condition. From this data one constructs an instanton on $\mathbf{R}^4$ which has rank N and topological charge k. This data can be extracted from the so-called "ADHM quiver", see figure **??**. More precisely, this data determines a representation of the ADHM quiver—roughly, the vector spaces V, W label the nodes (there is a framed and an unframed node) and the morphisms are labeled by the edges. The word 'quiver' simply refers to a directed graph. The maneuver of associating to a quiver the above linear data will be explained in this course.

To connect to the Hilbert scheme on $\mathbf{A}^2$ one should look at rank one instantons on $\mathbf{R}^4$. Strictly speaking, there are no instantons, but a slight variant of the above construction in terms of the ADHM quiver returns the Hilbert scheme. Roughly speaking, the moduli space of 'non-commutative' rank one instantons of charge k can be identified with the Hilbert scheme of k points on $\mathbf{A}^2$.

On a more general class of non-compact four-manifolds which are *asymptotically locally Euclidean* (ALE) there is a description of instantons in terms of more general quadratic equations also defined on some space of finite-dimensional matrices [KN90; Nak94]. The corresponding moduli spaces can be described in terms of a more general class of quivers whose underlying graphs are the Dynkin graphs of type ADE. This is not an accident: there is a classification due to Kronheimer of four-dimensional ALE spaces: they resolutions of singularities of the form $\mathbf{C}^2/\Gamma$ where $\Gamma \subset SU(2)$ is a finite subgroup. Finite subgroups of $SU(2)$ fall under the same ADE classification as finite simple Lie groups. For this reason, sometimes $\mathbf{C}^2/\Gamma$ is referred to as a ADE singularity. The relationship between these related classifications follows from the fact that both structures are governed by the combinatorics of the simply laced Dynkin diagrams.

Even if we forget the gauge theoretic origin, associated to any quiver is a moduli space of representations called the *Nakajima quiver variety*. These will be the main geometric objects we are concerned with in this course. In many cases, there are independent, algebro-geometric descriptions of these moduli spaces in terms of sheaves on complex varieties of dimension two. For example, in the ADHM case, it corresponds to the moduli space of torsion-free sheaves on $\mathbb{P}^2$ which are of rank N, framed at $\infty \in \mathbb{P}^2$, and have second Chern class equal to k.

1.2. Infinite-dimensional Lie algebras

Cohomology is one of the fundamental invariants of a space. In this course we will give a description of the cohomology of the Hilbert scheme and of more general quiver varieties. For the Hilbert scheme we will work out a beautiful formula for the generating function of the Poincaré polynomial derived originally by Göttsche [Göt90]. This generating function is of the form

$$\sum_{n\geq 0} q^n P_{\mathrm{Hilb}_n(X)}(t). \tag{2}$$

Here $P_Y(t) = \sum_n t^n \cdot (\dim H^n(Y))$ is the Poincaré polynomial of a space Y.

Amazingly, this generating function matches with an expression for the character of a representations for a certain infinite-dimensional algebra called the Heisenberg algebra. The Heisenberg algebra $\mathfrak{heis}$ is a central extension of the abelian Lie algebra of Laurent polynomials in a single variable

$$\mathbf{C} \to \mathfrak{heis} \to \mathbf{C}((z)). \tag{3}$$

It has irreducible representations labeled by a 'level' and a highest weight. One of the main results of Nakajima [Nak97] and Grojnowski [Gro96] is that for X an algebraic surface the direct sum of the homologies of Hilbert schemes

$$\oplus_{n\geq 0} H_\bullet\left(\mathrm{Hilb}_n(X)\right) \tag{4}$$

is a representation for $\mathfrak{heis}$. Moreover, the action of the Heisenberg algebra can be constructed in a completely geometric way and each $H_\bullet(\mathrm{Hilb}_n(X))$ is a weight space. Furthermore, one can actually argue that a richer structure is present on the direct sum above.

A *vertex algebra* is a structure present in two-dimensional conformal field theory (CFT). It is the algebraic structure carried by the so-called 'local operators' of a holomorphic two-dimensional CFT. The direct sum above turns out to be a vertex algebra in a totally geometric way.

For the case of higher ranks or more general quivers there is a similar picture. Here, the Heisenberg algebra is replaced by a Lie algebras of affine Kac–Moody type [Kac90].

1.3. Connection to string theory

A natural question to ask is for an 'explanation' for the relationship between CFT and the Hilbert scheme or instanton moduli spaces. One potential answer can be found in *string theory*. The Hilbert scheme of points and its higher rank generalizations take part in a rich collection of dualities in string theory such as the correspondence of Alday, Gaiotto, and Tachikawa [AGT10].

There is an infamous six-dimensional supersymmetric quantum field theory which can be defined for any Lie algebra of type ADE, just as in the classification of ALE spaces. From the point of view of string theory, one can think about the theory as obtained from 'compactifying' string theory on an ALE space. Though, no rigorous description of this theory exists, physicists are still able to glean useful information from this setup.

A setup relevant to the above discussion is to consider the six-dimensional theory on a product of manifolds of the form

$$\Sigma \times X \tag{5}$$

where Σ is a Riemann surface and X is a complex two-dimensional surface. The 'compactification' in the X direction yields a two-dimensional CFT whose states bear a close relationship to the cohomology of moduli spaces we will be considering. The remaining Σ-direction exhibits the rich structure of a CFT that we alluded to above.

Bibliography

[AGT10] L. F. Alday, D. Gaiotto, and Y. Tachikawa. "Liouville correlation functions from four-dimensional gauge theories". *Lett. Math. Phys.* 91.2 (2010), pp. 167–197. URL: https://doi.org/10.1007/s11005-010-0369-5.

[Ati+78] M. F. Atiyah, N. J. Hitchin, V. G. Drinfeld, and Y. I. Manin. "Construction of instantons". *Phys. Lett. A* 65.3 (1978), pp. 185–187. URL: https://doi.org/10.1016/0375-9601(78)90141-X.

[Göt90] L. Göttsche. "The Betti numbers of the Hilbert scheme of points on a smooth projective surface". *Math. Ann.* 286.1-3 (1990), pp. 193–207. URL: https://doi.org/10.1007/BF01453572.

[Gro96] I. Grojnowski. "Instantons and affine algebras. I. The Hilbert scheme and vertex operators". *Math. Res. Lett.* 3.2 (1996), pp. 275–291. URL: https://doi.org/10.4310/MRL.1996.v3.n2.a12.

[Kac90] V. G. Kac. *Infinite-dimensional Lie algebras*. Third. Cambridge University Press, Cambridge, 1990, pp. xxii+400. URL: https://doi.org/10.1017/CBO9780511626234.

[Kir16] A. Kirillov Jr. *Quiver representations and quiver varieties*. Vol. 174. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2016, pp. xii+295. URL: https://doi.org/10.1090/gsm/174.

[KN90] P. B. Kronheimer and H. Nakajima. "Yang-Mills instantons on ALE gravitational instantons". *Math. Ann.* 288.2 (1990), pp. 263–307. URL: https://doi.org/10.1007/BF01444534.

[Nak94] H. Nakajima. "Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras". *Duke Math. J.* 76.2 (1994), pp. 365–416. URL: https://doi.org/10.1215/S0012-7094-94-07613-8.

[Nak97] H. Nakajima. "Heisenberg algebra and Hilbert schemes of points on projective surfaces". *Ann. of Math. (2)* 145.2 (1997), pp. 379–388. URL: https://doi-org.ezproxy.bu.edu/10.2307/2951818.

[Nak99] H. Nakajima. *Lectures on Hilbert schemes of points on surfaces*. Vol. 18. University Lecture Series. American Mathematical Society, Providence, RI, 1999, pp. xii+132. URL: https://doi.org/10.1090/ulect/018.