LECTURE 1

The cohomology of the Hilbert scheme

Our goal in the next two lectures is to characterize the cohomology of the Hilbert scheme. Specifically, we will compute a generating function for the Poincaré polynomial. The main result is.

THEOREM 1.0.1. *Let $P_n(t)$ be the Poincaré polymomial of* $\mathrm{Hilb}_n(\mathbf{C}^2)$. *Then there is an equality*

$$\sum_{n\geq 1} q^n P_n(t) = \prod_{m\geq 1} \frac{1}{1-t^{2m-2}q^m}. \tag{1}$$

1.1. MORSE FUNCTIONS

Recall that a critical point of a smooth function $f\colon X \to \mathbf{R}$ is a point $x_0 \in X$ such that $\mathrm{d}f|_{x_0} = 0$. If x_0 is such a critical point then define the Hessian of f at x_0 to be the linear map

$$\mathrm{Hess}(f, x_0)\colon T_{x_0}M \times T_{x_0}M \to \mathbf{R} \tag{2}$$

given by

$$\mathrm{Hess}(f, x_0)(X_0, Y_0) = (XY)(x_0) \tag{3}$$

where X, Y are vector fields which evaluate to X_0, Y_0 respectively at x_0. An easy computation shows that the Hessian is symmetric and does not depend on the vector fields extending v, w chosen.

If we choose local coordinates (x_i) near x_0 such that $x_i(x_0) = 0$ then

$$\mathrm{Hess}(f, x_0)(X, Y) = \sum_{i,j} h_{ij} X^i Y^j, \qquad h_{ij} = (\partial_i \partial_j f)(x_0). \tag{4}$$

Here X^i, Y^j are the components of X, Y along ∂_i, ∂_j. Of course, the Hessian appears in the Taylor expansion of f near x_0 as

$$f(x) = f(x_0) + \frac{1}{2}\sum_{i,j} h_{ij} x_i x_j + \mathcal{O}(x^3). \tag{5}$$

Definition 1.1.1. A function $f\colon M \to \mathbf{R}$ is called a ***Morse function*** if all of its critical points result in a non-degenerate Hessian. In other words, all critical points of f are non-degenerate.

The utility of Morse functions lie in their rigidity.

THEOREM 1.1.2 ([**Morse**]). *Let $n = \dim M$ and suppose $f\colon M \to \mathbf{R}$ and let p_0 be a non-degenerate critical point of f. Then there exists a coordinitized open neighborhood $(x_i)\colon U \to \mathbf{R}^n$ of p_0 such that*

$$x_i(p_0) = 0, \quad f|_U(x) = f(p_0) + \frac{1}{2}\sum_{i,j} h_{ij}x_i x_j. \tag{6}$$

Recall that if B is a symmetric, non-degenerate, bilinear functional on a vector space V we can assign an integer called its ***index***. There is a basis $\{e_i\}$ for V such that

$$B(v,v) = -(|v^1|^2 + \cdots + |v^d|^2) + (|v^{d+1}|^2 + \cdots). \tag{7}$$

The index is defined to be the integer d. This integer is independent of the basis chosen.

Let $d(f,p)$ be the index of the Hessian of a function f at a critical point p. The ***Morse polynomial*** is

$$P_f(t) \overset{\text{def}}{=} \sum_p t^{d(f,p)} = \sum_{d\geq 0} \mu_f(d) t^{\lambda}. \tag{8}$$

Here the first sum is over all critical points of f. The sum on the right hand side simply reorganizes all terms so that the so-called Morse number $\mu_f(d)$ is the number of critical points of index d.

Perhaps the most direct relationship between Morse theory and topology is through the famous Morse inequalities. Consider the ring $\mathbf{Z}((t))$ of Laurent polynomials with integral coefficients. Define an ordering on this ring as follows. For $f, g \in \mathbf{Z}((t))$ we say $f \geq g$ if there exists $Q \in \mathbf{Z}((t))$ with non-negative coefficients such that

$$f(t) = g(t) + (1+t)Q(t). \tag{9}$$

Lemma 1.1.3. *Suppose that $A^\bullet \to B^\bullet \to C^\bullet$ is a short exact sequence of cochain complexes and let P_A, P_B, P_C be the corresponding Poincaré polynomials. Then*

$$P_B(t) \leq P_A(t) + P_C(t) \tag{10}$$

using this ordering.

The Morse inequalities relate the Morse polynomial to the Poincaré polynomial of the underlying manifold

$$P_X(t) \overset{\text{def}}{=} \sum_k (-1)^k \dim H^k(M) \tag{11}$$

THEOREM 1.1.4. *Suppose that $f\colon M \to \mathbf{R}$ is a Morse function where M is a compact manifold of dimension n and let $P_f(t)$ be the Morse polynomial defined above. Then*

$$P_M(t) \leq P_f(t) \tag{12}$$

A Morse function f is ***perfect*** if the Morse inequality is saturated $P_M(t) = P_f(t)$.

Example 1.1.5. Consider the sphere $S^n \subset \mathbf{R}^{n+1}$ with coordinates $(x_0, x_1, \ldots, x_n)$ satisfying $x_0^2 + \cdots + x_n^2 = 1$. The *height function* is

$$h_n\colon S^n \to \mathbf{R}, \quad (x_0, x_1, \ldots, x_n) \mapsto x_0. \tag{13}$$

This is a Morse function with two critical points $N = (1, 0, \dots, 0)$ and $S = (-1, 0, \dots, 0)$. The index of the northpole is $d(h, N) = n$ while the index of the southpole is $d(h, S) = 0$. Thus

$$P_f(t) = 1 + t^n = P_{S^n}(t) \tag{14}$$

and so h is perfect.

In fact, a Morse function whose indices are all even is necessarily perfect.

1.2. Morse theory for moment maps

Suppose that a torus T acts on a compact symplectic manifold (X, ω) with moment map $\mu \colon X \to \mathfrak{t}^*$. Let X^T be the fixed point set of the T-action. For any $\xi \in \mathfrak{t}$ consider the function

$$f \stackrel{\text{def}}{=} \langle \mu, \xi \rangle \colon X \to \mathbf{R}. \tag{15}$$

Critical points of f are easy to describe in terms of the T-action. Indeed, $x \in X$ is a critical point if and only if the vector field associated to ξ, denoted V_ξ, satisfies $V_\xi|_x = 0$. This is equivalent to the condition that

$$g \cdot x = x, \quad \text{for all } g \in \overline{\exp \mathbf{R}\xi}. \tag{16}$$

If ξ is chosen generically then $T = \overline{\exp \mathbf{R}\xi}$ and hence

$$\mathrm{Crit}(f) = X^T. \tag{17}$$

The fixed point set X^T decomposes into connected components as

$$X^T = \sqcup_{\nu \in I} C_\nu. \tag{18}$$

The main result that we will discuss next time is that in this situation the Morse function is perfect as long as X is compact and we have the following formula for the Poincaré polynomial

$$P_t(X) = \sum_\nu t^{d_\nu} P_t(C_\nu). \tag{19}$$