LECTURE 5

Moduli spaces of sheaves, I

Last time we showed that the Hilbert scheme of n points in $\mathbf{A}^2$ is non-singular and equivalent to the quotient of

$$\widetilde{H}_n^s = \{(X, Y, i) \mid [X, Y] = 0, \quad \text{and stability}\} \subset \operatorname{End}(\mathbf{C}^n)^{\oplus 2} \oplus \mathbf{C}^n. \tag{1}$$

by the natural $GL(n, \mathbf{C})$ action. Today we will wrap up this discussion with a computation of the dimension of $\operatorname{Hilb}_n(\mathbf{A}^2)$ and some examples of Hilbert schemes for small values of n. Then, we turn to a sheaf-theoretic description of the Hilbert scheme.

5.1. Dimension of the Hilbert scheme

For $(X, Y, i) \in \widetilde{H}_n^s$ let $(C^\bullet, \mathrm{d})$ be the following complex

$$\operatorname{End}(\mathbf{C}^n) \xrightarrow{\mathrm{d}_1} \operatorname{End}(\mathbf{C}^n)^{\oplus 2} \oplus \mathbf{C}^n \xrightarrow{\mathrm{d}_2} \operatorname{End}(\mathbf{C}^n) \tag{2}$$

where the first arrow is the derivative of the $GL(n, \mathbf{C})$ action

$$\mathrm{d}_1(A) = ([A, X], [A, Y], Ai) \tag{3}$$

and the second arrow is

$$\mathrm{d}_2(A, B, v) = [X, A] + [Y, B]. \tag{4}$$

Then the tangent space at (X, Y, i) is

$$\mathrm{T}_{(X,Y,i)} \operatorname{Hilb}_n(\mathbf{A}^2) \simeq H^1(C, \mathrm{d}). \tag{5}$$

We have already shown that the dimension of the cokernel of d_2 is n. By the stability condition we have $\ker \mathrm{d}_1 = 0$. This shows that $\dim H^1(C) = 2n$.

5.2. Examples

Let's consider some examples of $\operatorname{Hilb}_n(\mathbf{A}^2)$ for small n. For $n = 1$ we have $X = x, Y = y$ for some numbers $x, y \in \mathbf{C}$. Furthermore, the stability condition implies that $i \neq 0$. Using the $\mathbf{C}^\times$-action we can assume that $i = 1$. The ideal corresponding to the pair x, y is

$$I = \{f(z_1, z_2) \in \mathbf{C}[z_1, z_2] \mid f(x, y) = 0\}. \tag{6}$$

This is simply the maximal idea corresponding to $(x, y) \in \mathbf{A}^2$. Thus $\operatorname{Hilb}_1(\mathbf{A}^2) = \mathbf{A}^2$.

Next we look at $n = 2$. Then X, Y are 2×2 matrices. Suppose that at least one of X, Y have distinct eigenvalues. Since $[X, Y] = 0$ we can assume that

$$X = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix} \tag{7}$$

with $(x_1, y_1) \neq (x_2, y_2)$. By the stability condition we can take

$$i(1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{8}$$

The corresponding ideal is

$$I = \{f(z_1, z_2) \in \mathbf{C}[z_1, z_2] \mid f(x_i, y_i) = 0\}, \tag{9}$$

which corresponds to two distinct points in $\mathbf{A}^2$. Thus away from the diagonal in $\mathbf{A}^2 \times \mathbf{A}^2$ the Hilbert scheme agrees with $S^2\mathbf{A}^2$.

The interesting stuff happens when we assume that X, Y each have one eigenvalue. We cannot assume that X, Y are both diagonalizable as this violates the stability condition. Thus, we have

$$X = \begin{pmatrix} x & a \\ 0 & x \end{pmatrix}, \quad Y = \begin{pmatrix} y & b \\ 0 & y \end{pmatrix} \tag{10}$$

for some $(a, b) \in \mathbf{A}^2 - 0$. In this basis we can assume that

$$i(1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{11}$$

The corresponding ideal is

$$I = \left\{ f(z_1, z_2) \in \mathbf{C}[z_1, z_2] \mid f(x, y) = \left(a\frac{\partial f}{\partial z_1} + b\frac{\partial f}{\partial z_2} \right)(x, y) = 0 \right\}. \tag{12}$$

This corresponds to two infinitesimally close points in $\mathbf{A}^2$ at (x, y) which point to each other in the direction of the vector field $a\frac{\partial}{\partial z_1} + b\frac{\partial}{\partial z_2}$.

5.3. TORSION-FREE SHEAVES

A quasi-coherent sheaf $\mathcal{F}$ on an algebraic variety X is ***torsion-free*** if for every affine open subset $U \subset X$ the space of local sections $\mathcal{F}(U)$ is torsion-free as a module over the ring of functions $\mathcal{O}(U)$ on U. That is, for ever nonzero section $s \in \mathcal{F}(U)$ and nonzero function $f : U \to \mathbf{C}$ one has $f \cdot s \neq 0$. A typical example of a torsion-free sheaf is the sheaf of sections of a vector bundle; the condition of being a locally free implies torsion-free. We will mostly be concerned with coherent torsion-free sheaves.

For any quasi-coherent sheaf $\mathcal{F}$ there is a canonical morphism

$$\mathcal{F} \to (\mathcal{F}^\vee)^\vee = \mathcal{F}^{\vee\vee} \tag{13}$$

where $\mathcal{F}^\vee = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ is the dual sheaf.[1] The main technical result about torsion-free sheaves that we will use is the following.

THEOREM 5.3.1 (**[??]**). *Let X be a non-singular algebraic variety and suppose $\mathcal{F}$ is a coherent torsion-free sheaf on X. Then:*

[1] A quasi-coherent sheaf which is isomorphic to its double dual is called reflexive.

- *there exists a Zariski open set $U \subset X$ of codimension ≥ 2 such that $\mathcal{F}|_U$ is locally free.*
- *If $\dim X = 2$ then the sheaf $\mathcal{F}^{\vee\vee}$ is locally free of finite rank and the morphism $\mathcal{F} \to \mathcal{F}^{\vee\vee}$ is injective. Restriction of this morphism to U results in an isomorphism $\mathcal{F}|_U \xrightarrow{\sim} \mathcal{F}^{\vee\vee}|_U$.*

By the second item we see that any torsion-free sheaf is a subsheaf of a coherent locally free sheaf. Consider the dual $\mathcal{F}^\vee$ and write this as the quotient of a locally free sheaf $\mathcal{E}$. Then $\mathcal{F}^{\vee\vee}$ is a subsheaf of $\mathcal{E}^\vee$ which is still locally free.

Example 5.3.2. Suppose that $\mathcal{F}$ is a rank one torsion-free sheaf on a surface S. Since $\mathcal{F}^{\vee\vee}$ is locally free it is a line bundle.

Example 5.3.3. Suppose that $J \in \mathrm{Hilb}_n(X)$ where X is any affine variety (not necessarily of dimension two). Let $\mathcal{F}_J$ be the corresponding ideal sheaf of $\mathcal{O}_X$ which satisfies $\Gamma(X, \mathcal{F}_J) = J$. Then $\mathcal{F}_J$ is torsion-free and $\mathcal{O}/\mathcal{F}_J$ is a torsion sheaf

$$\Gamma(X, \mathcal{O}_X/J) = \mathbf{C}[X]/J. \tag{14}$$

Example 5.3.4. Consider the ideal sheaf of the point $0 \in \mathbf{A}^2$. This sheaf is torsion-free but not locally free.

Example 5.3.5. Consider the morphism of sheaves on $\mathbf{A}^2$

$$\phi\colon \mathcal{O} \to \mathcal{O}^{\oplus 2} \tag{15}$$

defined by $\Phi(f) = (z_1 f, z_2 f)$. Then ϕ is injective and its image is

$$\mathrm{im}(\phi) = \{(f_1, f_2) \mid z_1 f_2 = z_2 f_1\} \subset \mathcal{O}^{\oplus 2}. \tag{16}$$

We claim that

$$\mathcal{F} \stackrel{\mathrm{def}}{=} \operatorname{coker} \phi \tag{17}$$

is a torsion-free sheaf. Indeed let $\psi\colon \mathcal{O}^{\oplus 2} \to \mathcal{O}$ be $\psi(g_1, g_2) = z_2 g_1 - z_1 g_2$. Then $\ker \psi = \mathrm{im}\, \phi$ so

$$\mathcal{F} \simeq \mathrm{im}\, \psi = \{f \in \mathcal{O} \mid f(0,0) = 0\}. \tag{18}$$

By this last equivalence we see that $\mathcal{F}$ is isomorphic to a subsheaf of $\mathcal{O}$.

More generally one has the following. Let V be a finite dimensional vector space and $A_1, A_2 \in \mathrm{End}(V)$. Denote by $\mathcal{V} = \mathcal{O} \otimes V$ the trivial vector bundle on $\mathbf{A}^2$ with fiber V. Define

$$\phi\colon \mathcal{V} \to \mathcal{V} \otimes \mathbf{C}^2 \tag{19}$$

by $s \mapsto ((A_1 - z_1)s, (A_2 - z_2)s)$.

Lemma 5.3.6. *The sheaf* $\operatorname{coker} \phi$ *is torsion-free.*