LECTURE 6

Moduli spaces of sheaves, II

We are heading towards a definition of the Hilbert scheme in terms sheaves.

6.1. Chern classes

Let X be a smooth algebraic variety over $\mathbf{C}$, which you are free to think of as just complex manifold. The jth Chern class of a complex vector bundle E over X is an element

$$c_j(E) \in H^{2j}(X; \mathbf{R}). \tag{1}$$

The total Chern class is usually denoted

$$c(E) = \sum_{j \geq 0} c_j(E) \in H^{2\bullet}(X; \mathbf{R}), \tag{2}$$

or its one parameter version

$$c_t(E) = \sum_{j \geq 0} t^j c_j(E) \in H^{2\bullet}(X; \mathbf{R})[t]. \tag{3}$$

The Chern classes are determined by the following axioms.

- *The zeroeth Chern class*. For any bundle $E \to X$ one has $c_0(E) = 1$.
- *Naturality*. For any bundle $E \to X$ and smooth map $f\colon Y \to X$ one has

$$c(f^*E) = f^*c(E) \in H^{2\bullet}(X; \mathbf{R}). \tag{4}$$

- *Whitney sum.* For a finite collection of bundles E_i one has

$$c(\oplus_i E_i) = \prod_i c(E_i). \tag{5}$$

- *Normalization.* Let $\mathcal{O}(1)$ be the dual of the tautological line bundle over $\mathbf{CP}^1$. Then

$$\int_{\mathbf{CP}^1} c_1(\mathcal{O}(1)) = 1. \tag{6}$$

We will need to extend the definition of Chern classes to coherent sheaves. Let $\mathrm{Coh}(X)$ be the category of coherent sheaves on X and let $\mathrm{Vect}(X) \subset \mathrm{Coh}(X)$ be the subcategory of locally free coherent sheaves. This subcategory is equivalent to the category of holomorphic vector bundles on X; the equivalence is obtained by taking the sheaf of holomorphic sections of a given holomorphic vector bundle. Both $\mathrm{Coh}(X)$ and $\mathrm{Vect}(X)$ are abelian categories.

Construction 6.1.1. Given any abelian category $\mathcal{A}$ we can look at the free abelian group $\mathbf{Z}[\mathcal{A}]$ which is generated by the isomorphism classes of objects of $\mathcal{A}$. Given a short exact sequence

$$0 \to A \to B \to C \to 0 \tag{7}$$

in $\mathcal{A}$ we can form the element

$$-[A] + [B] - [C] \in \mathbf{Z}[\mathcal{A}]. \tag{8}$$

Let $E(\mathcal{A})$ be the subgroup of $\mathbf{Z}[\mathcal{A}]$ generated by elements of this form. The ***Grothendieck group*** of the abelian category $\mathcal{A}$ is defined as the quotient group

$$K_0(\mathcal{A}) \stackrel{\text{def}}{=} \mathbf{Z}[\mathcal{A}]/E(\mathcal{A}). \tag{9}$$

By definition, if (7) is a short exact sequence then we have the relation

$$[B] = [A] + [C] \tag{10}$$

in $K_0(\mathcal{A})$.

If $\mathcal{A}_0 \subset \mathcal{A}$ is an additive (not necessarily full) subcategory which is closed under extensions then the above definition endows $K_0(\mathcal{A}_0)$ also with the structure of an abelian group. Such an $\mathcal{A}_0$ is called an ***exact*** category.

We apply this construction to the situation

$$\mathcal{A}_0 = \mathrm{Vect}(X) \subset \mathrm{Coh}(X) = \mathcal{A}. \tag{11}$$

Notice that tensor product endows both $K_0(X) = K_0(\mathrm{Vect}(X))$ with the structure of a commutative ring.

Lemma 6.1.2. *Let X be a smooth complex variety or a complex manifold.*

1. *The subring $E(\mathrm{Vect}(X)) \subset \mathbf{Z}[\mathrm{Vect}(X)]$ is an ideal, and therefore $K_0(X)$ has the structure of a commutative ring with unit given by the trivial rank one vector bundle.*
2. *The group $K_0(\mathrm{Coh}(X))$ is naturally a module for $K_0(X)$.*
3. *The embedding $\mathbf{Z}[\mathrm{Vect}(X)] \hookrightarrow \mathbf{Z}[\mathrm{Coh}(X)]$ determines a group homomorphism*

$$i\colon K_0(X) \to K_0(\mathrm{Coh}(X)). \tag{12}$$

By the axioms of Chern classes above, we see that the total Chern class defines a group homomorphism

$$c\colon K_0(X) \to H^\bullet(X). \tag{13}$$

An immediate consequence of this is a slightly more general version of the Whitney sum axiom. If we have any exact sequence of vector bundles

$$0 \to E' \to E \to E'' \to 0 \tag{14}$$

then

$$c_t(E) = c_t(E') \cdot c_t(E''). \tag{15}$$

Remark 6.1.3. In fact, there is a more refined relationship between $K_0(X)$ and the cohomology of X.

The Chern character of a complex vector bundle $E \to X$ is an element

$$\mathrm{ch}(E) \in H^{2\bullet}(X; \mathbf{R}) \tag{16}$$

defined formally as follows. Suppose that ξ_i are constants and x is a formal variable such that

$$\sum_i c_i(E)x^i = \prod_i (1+\xi_i x). \tag{17}$$

Then the Chern character is defined by

$$\mathrm{ch}(E) = \sum_i e^{\xi_i}. \tag{18}$$

The Chern character enjoys a similar sum rule $\mathrm{ch}(\oplus_i E_i) = \sum_i \mathrm{ch}(E_i)$ and also a product identity

$$\mathrm{ch}(\otimes_i E_i) = \prod_i \mathrm{ch}(E_i). \tag{19}$$

Immediately, then, we see that the Chern character defines a ring homomorphism

$$\mathrm{ch}\colon K_0(X) \to H^\bullet(X). \tag{20}$$

Now, we can see how to extend Chern classes to coherent sheaves. Given a coherent sheaf $\mathcal{F}$ on a smooth projective algebraic variety over $\mathbf{C}$ there exists a locally free resolution of $\mathcal{F}$ (that is, a resolution by vector bundles) of the form

$$0 \to \mathcal{E}_{-n} \to \mathcal{E}_{-n+1} \cdots \to \mathcal{E}_{-1} \to \mathcal{E}_0 \to \mathcal{F} \to 0. \tag{21}$$

In the case of a general complex manifold such a resolution is only guaranteed to exist locally. Using such a resolution we define

$$c(\mathcal{F}) \overset{\text{def}}{=} \sum_i (-1)^i c(\mathcal{E}_i) \in H^\bullet(X). \tag{22}$$

One can show that this definition does not depend on the resolution.

This construction can be refined to providing an inverse j to the ring homomorphism $i\colon K_0(X) \to K_0(\mathrm{Coh}(X))$ by the formula

$$j([\mathcal{F}]) = \sum_i (-1)^i [\mathcal{E}_i]. \tag{23}$$

The proof of the fact that these homomorphisms are inverses to each other is outside of the scope of these notes.

An important computational tool we will use, but not spend time providing background on, is the Grothendieck–Riemann–Roch theorem. This very powerful result is a generalization of the Hirzebruch–Riemann–Roch theorem in the context of holomorphic vector bundles and complex manifolds.

Suppose that $\mathcal{E}$ is a coherent sheaf of X and that $f\colon X \to Y$ is a proper map between smooth quasi-projective varieties. The Grothendieck–Riemann–Roch theorem presents a formula for the characteristic classes of $f_!\mathcal{E}$ as

$$\mathrm{ch}(f_!\mathcal{E}) \cdot \mathrm{Td}(Y) = f_* \left(\mathrm{ch}(\mathcal{E}) \cdot \mathrm{Td}(Y)\right) \tag{24}$$

where

- ch is the Chern character as above which admits an expansion like

$$\mathrm{ch} = \mathrm{rk} + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{3!}(c_1^3 - 3c_1c_2 + 3c_3) + \cdots . \tag{25}$$

- td is the Todd class which admits an expansion like

$$\mathrm{td} = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1 c_2 + \cdots \tag{26}$$

- $f_! = \sum(-1)^i \mathbf{R}^i f_* \colon K_0(X) \to K_0(Y)$ is the higher direct image or pushforward map in K-theory.
- $f_* \colon H^\bullet(X) \to H^\bullet(Y)$ is the pushforward map in cohomology.

Example 6.1.4. Suppose that $Y = pt$ and that $\mathcal{E}$ is a vector bundle $E \to X$. Then

$$\mathrm{ch}(f_! \mathcal{E}) = \chi(X, E) \tag{27}$$

is the holomorphic Euler characteristic. In this case the map f_* in cohomology is of degree $-2n$, so the theorem implies the Hirzebruch–Riemann-Roch theorem

$$\chi(X, E) = [\mathrm{ch}(E)\,\mathrm{td}(X)]_{2n}\,. \tag{28}$$

Example 6.1.5. Suppose that we have a closed embedding $i \colon Y \hookrightarrow Z$ with corresponding ideal sheaf $\mathcal{I}_Y$. From the short exact sequence

$$0 \to \mathcal{I}_Y \to \mathcal{O}_X \to \mathcal{O}_Y \to 0 \tag{29}$$

we can see that

$$c_k(\mathcal{I}_Y) = (-1)^k (k-1)! [Y] \tag{30}$$

where k is the codimension of Y in X and $[Y]$ is the fundamental class of Y. In particular if X is d-dimensional and Y is zero-dimensional with n-connected components then

$$c_d(\mathcal{I}_Y) = (-1)^d n (d-1)!. \tag{31}$$

6.2. Torsion-free sheaves on surfaces

Recall that if $\mathcal{E}$ is a torsion-free sheaf on a surface X then the natural map $\mathcal{E} \to \mathcal{E}^{\vee\vee}$ is injective. In particular, there is an induced short exact sequence

$$0 \to \mathcal{E} \to \mathcal{E}^{\vee\vee} \to \mathcal{Q} \to 0 \tag{32}$$

The cokernel sheaf $\mathcal{Q}$ has the property that its support is zero-dimensional.

Consider projective space $\mathbf{P}^2$ and let $\ell_\infty \subset \mathbf{P}^2$ be the line

$$\ell_\infty = \{(0 \colon z_2 \colon z_3)\} \subset \mathbf{P}^2. \tag{33}$$

Definition 6.2.1. Let $\mathcal{E}$ be a torsion-free sheaf on $\mathbf{P}^2$ of rank r. A ***framing*** is an isomorphism $\Phi \colon \mathcal{E}|_{\ell_\infty} \xrightarrow{\simeq} \mathcal{O}_{\ell_\infty}^{\oplus r}$. Denote a framed sheaf by a pair $(\mathcal{E}, \Phi)$.

For a torsion-free sheaf the only topological invariant is its second Chern class, which follows from the following lemma.

Lemma 6.2.2. *Suppose $\mathcal{E}$ is a torsion-free sheaf on $\mathbf{P}^2$ which admits a framing. Then $c_1(\mathcal{E}) = 0$.*

Proof. From the short exact sequence (32) we see that $c_1(\mathcal{E}) = c_1(\mathcal{E}^{\vee\vee})$. By the framing condition, the support of the cokernel sheaf $\mathcal{Q}$ cannot intersect ℓ_∞, thus $\mathcal{E}|_{\ell_\infty} \simeq \mathcal{E}^{\vee\vee}|_{\ell_\infty}$. Finally, since $\mathcal{E}^{\vee\vee}$ is a line bundle we have $0 = c_1(\mathcal{E}^{\vee\vee}|_{\ell_\infty}) = c_1(\mathcal{E}^{\vee\vee})|_{\ell_\infty}$ which implies $c_1(\mathcal{E}^{\vee\vee}) = 0$. □

A map of framed sheaves

$$F\colon (\mathcal{E},\Phi) \to (\mathcal{E}',\Phi') \tag{34}$$

is a map of sheaves $F\colon \mathcal{E} \to \mathcal{E}'$ which intertwines the framings.

Definition 6.2.3. Let $\mathcal{M}^{fr}(r,n)$ be the moduli space of framed sheaves $(\mathcal{E},\Phi)$ on $\mathbf{P}^2$ of rank r and $c_2(\mathcal{E}) = n$. As a set this is the set of isomorphism classes

$$\{[(\mathcal{E},\Phi)] \mid \mathrm{rk}(E) = r, \quad c_2(\mathcal{E}) = n\}. \tag{35}$$

This only really defines $\mathcal{M}^{fr}(n,r)$ as a set, but it can be shown that it can be given the structure of a scheme [**NakajimaBook**]. The rank one case is especially relevant to the previous lectures.

Proposition 6.2.4. *There is an isomorphism*

$$\mathcal{M}^{fr}(1,n) \simeq \mathrm{Hilb}_n(\mathbf{A}^2). \tag{36}$$

PROOF. Let $\mathcal{E}$ be a rank one torsion-free sheaf of second Chern class n. By the framing condition we have an embedding

$$\mathcal{E} \hookrightarrow \mathcal{E}^{\vee\vee} \simeq \mathcal{O}_{\mathbf{P}^2}. \tag{37}$$

We have already pointed out that the quotient sheaf $\mathcal{Q} = \mathcal{E}^{\vee}/\mathcal{E}$ has zero-dimensional support away from $\ell_\infty \subset \mathbf{P}^2$ and satisfies

$$\dim \Gamma(\mathbf{P}^2 - \ell_\infty, \mathcal{Q}) = n. \tag{38}$$

This gives the correspondence

$$\mathcal{M}^{fr}(1,n) \xrightarrow{\simeq} \mathrm{Hilb}_n(\mathbf{P}^2 - \ell_\infty) \simeq \mathrm{Hilb}_n(\mathbf{A}^2) \tag{39}$$

□

Next time we will explain the so-called ADHM description of the moduli space $\mathcal{M}^{fr}(r,n)$ which is in the same spirit as the description of the Hilbert scheme in terms of matrices. From there we will discuss the symplectic structure on these moduli spaces.