LECTURE 7

Moduli spaces of sheaves, III

Today we will deduce the following description of the moduli space of torsion-free rank r sheaves. Let $H(r,n)$ be the affine subspace of

$$\mathrm{End}(\mathbf{C}^n)^{\oplus n} \oplus \mathrm{Hom}(\mathbf{C}^r, \mathbf{C}^n) \oplus \mathrm{Hom}(\mathbf{C}^n, \mathbf{C}^r) \tag{1}$$

consisting of tuples (X, Y, I, J) such that

- $[X, Y] + IJ = 0$.
- there exists no proper subspace $S \subset \mathbf{C}^n$ such that $X \cdot S \subset S$, $Y \cdot S \subset S$ and $\mathrm{im}\, I \subset S$.

We refer to the last item as the stability condition. There is an action of $GL(n, \mathbf{C})$ on $H(r,n)$ defined by

$$g \cdot (X, Y, I, J) = (gXg^{-1}, gYg^{-1}, gI, Jg^{-1}). \tag{2}$$

THEOREM 7.0.1 (Barth). *Let $\mathcal{M}(r,n)$ denote the moduli space of torsion-free sheaves on $\mathbf{P}^2$ of rank r and $c_2 = n$. There is an isomorphism*

$$\mathcal{M}(r,n) \simeq H(r,n)/GL(n,\mathbf{C}). \tag{3}$$

Like the Hilbert scheme, there is a functorial definition of $\mathcal{M}(r,n)$ which makes its scheme structure manifest. The $GL(n,\mathbf{C})$ action on $H(r,n)$ is free so also like in the case of the Hilbert scheme $H(r,n)/GL(n,\mathbf{C})$ agrees with the affine GIT quotient. The above bijection of sets can be enhanced to an isomorphism of affine schemes.

This lecture closely follows the arguments in §2 of [Nak99].

7.1. TECHNICAL LEMMA

Suppose that $\mathcal{E}$ is any sheaf on $\mathbf{P}^2$ and let $\mathcal{E}(k) = \mathcal{E} \otimes_{\mathcal{O}} \mathcal{O}(k)$. Also let $\mathcal{Q}$ denote the rank two vector bundle

$$\mathcal{Q} = \mathcal{T}_{\mathbf{P}^2}(-1) \tag{4}$$

where $\mathcal{T}_{\mathbf{P}^2}$ is the tangent bundle. In the following we assume that $\mathcal{E}$ is a torsion-free sheaf of rank r with $c_2(\mathcal{E}) = n$. We assume that $\mathcal{E}$ is framed at the line $\ell_\infty = \{[0 : z_1 : z_2]\} \subset \mathbf{P}^2$.

Lemma 7.1.1. *For $p = 1, 2$ and $q = 0, 2$ one has*

$$H^q(\mathbf{P}^2, E(-p)) = 0 \tag{5}$$

and

$$H^q(\mathbf{P}^2, E(-1) \otimes \mathcal{Q}). \tag{6}$$

PROOF. Tensoring the short exact sequence

$$0 \to \mathcal{O}(-1) \to \mathcal{O} \to \mathcal{O}_{\ell_\infty} \to 0, \tag{7}$$

with $\mathcal{E}(-p)$ yields the short exact sequence

$$0 \to \mathcal{E}(-p-1) \to \mathcal{E}(-p) \to \mathcal{E}(-p)|_{\ell_\infty} \to 0. \tag{8}$$

Since $\mathcal{E}|_{\ell_\infty} \simeq \mathcal{O}_{\ell_\infty}^{\oplus r}$ we have

$$H^0(\mathbf{P}^2, \mathcal{E}(-p)|_{\ell_\infty}) = H^1(\mathbf{P}^2, \mathcal{E}(-p)|_{\ell_\infty}) = 0. \tag{9}$$

From the resulting long exact sequence in cohomology we obtain that

$$H^0(\mathbf{P}^2, \mathcal{E}(-p-1)) \simeq H^0(\mathbf{P}^2, \mathcal{E}(-p)) \tag{10}$$

for $p \geq 1$ and

$$H^2(\mathbf{P}^2, \mathcal{E}(-p-1)) \simeq H^2(\mathbf{P}^2, \mathcal{E}(-p)) \tag{11}$$

for $p \leq 1$.

Recall that since $\mathcal{E}$ is torsion-free we have that $\mathcal{E}^{\vee\vee}$ is locally free and that the canonical map $\mathcal{E} \hookrightarrow \mathcal{E}^{\vee\vee}$ is injective. Hence there is an injection $H^0(\mathbf{P}^2, \mathcal{E}(-p)) \hookrightarrow H^0(\mathbf{P}^2, \mathcal{E}^{\vee\vee}(-p))$. By Serre duality this means we have an inclusion

$$H^0(\mathbf{P}^2, \mathcal{E}(-p)) \hookrightarrow H^2(\mathbf{P}^2, (\mathcal{E}^{\vee\vee}(p-3)))^{\vee} \tag{12}$$

By the Serre vanishing theorem, for p large enough the right hand side is trivial. Combining with (10) we see that

$$H^0(\mathbf{P}^2, \mathcal{E}(-1)) \simeq H^0(\mathbf{P}^2, \mathcal{E}(-2)) \simeq \cdots \simeq 0. \tag{13}$$

Also by Serre vanishing together with (11) we see that

$$H^2(\mathbf{P}^2, \mathcal{E}(-2)) \simeq H^2(\mathbf{P}^2, \mathcal{E}(-1)) \simeq H^2(\mathbf{P}^2, \mathcal{E}) \simeq \cdots \simeq 0. \tag{14}$$

By a similar argument one obtains that $H^q(\mathbf{P}^2, \mathcal{Q} \otimes \mathcal{E}(-1))$ for $q = 0, 2$. □

7.2. THE MONADIC DESCRIPTION OF A TORSION-FREE SHEAF

Define the following vector spaces

- $V^{-1} = H^1(\mathbf{P}^2, \mathcal{E}(-2))$. Note that by the lemma above we have

$$\chi(\mathbf{P}^2, \mathcal{E}(-2)) = -\dim H^1(\mathbf{P}^2, \mathcal{E}(-2)). \tag{15}$$

  Also by Hirzebruch–Riemann–Roch

$$\begin{aligned} \chi(\mathbf{P}^2, \mathcal{E}(-2)) &= \int_{\mathbf{P}^2} \mathrm{ch}(\mathcal{E}(-2))\,\mathrm{td}(\mathbf{P}^2) \\ &= r \int_{\mathbf{P}^2} \mathrm{ch}(\mathcal{O}(-2))\,\mathrm{td}(\mathbf{P}^2) - \int_{\mathbf{P}^2} c_2(E) \\ &= r\chi(\mathbf{P}^2, \mathcal{O}(-2)) - n = -n. \end{aligned}$$

  Thus $\dim V^{-1} = n$, so $V^{-1} \simeq \mathbf{C}^n$.
- $V^0 = H^1(\mathbf{P}^2, \mathcal{E}(-1) \otimes \mathcal{Q}^{\vee})$. This vector space has dimension $2n + r$, so $V^0 \simeq \mathbf{C}^{2n+r}$.
- $V^1 = H^1(\mathbf{P}^2, \mathcal{E}(-1))$. This vector space has dimension n, so $V^1 \simeq \mathbf{C}^n$.

We won't prove the following lemma.

Lemma 7.2.1. *Let $\mathcal{E}$ be a framed torsion-free sheaf. Then, there is a complex of sheaves*

$$(\mathcal{V}^\bullet, \mathrm{d}) \tag{16}$$

with $\mathcal{V}^i = \mathcal{O}_{\mathbf{P}^2}(i) \otimes V^i$ such that $H^{-1} = H^1 = 0$ and

$$H^0(\mathcal{V}^\bullet, \mathrm{d}) \simeq \mathcal{E}. \tag{17}$$

Let $a = \mathrm{d}_{-1\to 0}$ and $b = \mathrm{d}_{0\to 1}$ be the differentials in the complex of sheaves $\mathcal{V}^\bullet$. We can view

$$a \in H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(1)) \otimes \mathrm{Hom}(V^{-1}, V^0)$$
$$b \in H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(1)) \otimes \mathrm{Hom}(V^0, V^1)$$

In particular, a, b must be of the form

$$a = a_0 z_0 + a_1 z_1 + a_2 z_2$$
$$b = b_0 z_0 + b_1 z_1 + b_2 z_2$$

for some linear maps $a_i \colon V^{-1} \to V^0$, $b_i \colon V^0 \to V^1$. Since $\mathrm{d}^2 = 0 \iff ba = 0$ we have the relations

$$b_0 a_0 = 0, \quad b_0 a_1 + b_1 a_0 = 0$$
$$b_1 a_1 = 0, \quad b_1 a_2 + b_2 a_1 = 0$$
$$b_2 a_2 = 0, \quad b_0 a_2 + b_2 a_0 = 0.$$

By the technical lemma we can form the exact sequence

$$0 \to \mathcal{V}^{-1} \to \ker b \to \mathcal{E} \to 0. \tag{18}$$

Consider the restriction of this complex of sheaves to the line at ∞

$$0 \to \mathcal{V}^{-1}|_{\ell_\infty} \xrightarrow{a|_{\ell_\infty}} \ker b|_{\ell_\infty} \to \mathcal{E}|_{\ell_\infty} \to 0. \tag{19}$$

Note that $a|_{\ell_\infty} = a_1 z_2 + a_2 z_2$, $b|_{\ell_\infty} = b_1 z_1 + b_2 z_2$. From the resulting long exact sequence in cohomology we see that

$$H^0(\ell_\infty, \ker b|_{\ell_\infty}) \simeq H^0(\ell_\infty, \mathcal{E}|_{\ell_\infty}) \simeq \mathcal{E}|_p$$
$$H^0(\ell_\infty, \ker b|_{\ell_\infty}) \simeq H^0(\ell_\infty, \mathcal{E}|_{\ell_\infty}) = 0$$

where $p \in \ell_\infty$. The first isomorphism implies that

$$W \stackrel{\text{def}}{=} H^0(\ell_\infty, \ker b|_{\ell_\infty}) \tag{20}$$

gives the trivialization of $\mathcal{E}$ at ∞, and hence the choice of basis of W gives the framing.

Similarly, we have the short exact sequence

$$0 \to \ker b|_{\ell_\infty} \to \mathcal{V}^0|_{\ell_\infty} \xrightarrow{b|_{\ell_\infty}} \mathcal{V}^1|_{\ell_\infty} \to 0. \tag{21}$$

The long exact sequence in cohomology yields another short exact sequence

$$0 \to W \to V^0 \xrightarrow{\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}} z_1 V^1 \oplus z_2 V^1 \to 0. \tag{22}$$

where we have identified $H^0(\ell_\infty, \mathcal{O}_{\ell_\infty}(1)) \simeq \mathbf{C}z_1 + \mathbf{C}z_2$. From this short exact sequence we see that $W = \ker b_1 \cap \ker b_2$. Applying similar arguments to the dual sequence we see that the map

$$(a_1, a_2)\colon \mathcal{O}_{\ell_\infty}(1) \otimes V^{-1} \simeq z_1 V^{-1} \oplus z_2 V^{-1} \to V^0 \tag{23}$$

is injective and that $a|_{\ell_\infty}$ is injective at each fiber.

Restricting the original short exact sequence to $[0 : 1 : 0] \in \ell_\infty$ we have a sequence

$$V^{-1} \xrightarrow{a_1} V^0 \xrightarrow{b_1} V^1 \tag{24}$$

and $\ker b_1 / \operatorname{im} a_1 = E_{[0:1:0]} \simeq \ker b_1 \cap \ker b_2 = W$. Thus $\operatorname{im} a_1 \cap \ker b_2 = \operatorname{im} a_1 \cap W = 0$ so that $b_2 a_1 \colon V^{-1} \to V^1$ is injective. Since V^{-1}, V^1 are of the same dimension, this map is an isomorphism.

Using this isomorphism we can identify $V = V^{-1} = V^1$ and the maps in the sequence

$$V \oplus V \xrightarrow{(a_1, a_2)} V^0 \xrightarrow{(b_1, b_2)^t} V \oplus V \tag{25}$$

with

$$a_1 = \begin{pmatrix} -\mathbb{1}_V \\ 0 \\ 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 0 \\ -\mathbb{1}_V \\ 0 \end{pmatrix} \tag{26}$$

and

$$b_1 = \begin{pmatrix} 0 & -\mathbb{1}_V & 0 \end{pmatrix}, \quad b_2 = \begin{pmatrix} \mathbb{1}_V & 0 & 0 \end{pmatrix}. \tag{27}$$

From $ba = 0$ we obtain the following form of the remaining components

$$a_0 = \begin{pmatrix} X \\ Y \\ J \end{pmatrix}, \quad b_0 = \begin{pmatrix} -Y & X & I \end{pmatrix}. \tag{28}$$

Here $[X, Y] + IJ = 0$.

This gives us the following monadic description of $\mathcal{E}$.

$$V \otimes \mathcal{O}_{\mathbf{P}^2}(-1) \xrightarrow{\ a\ } \begin{pmatrix} V \\ V \\ W \end{pmatrix} \otimes \mathcal{O}_{\mathbf{P}^2} \xrightarrow{\ b\ } V \otimes \mathcal{O}_{\mathbf{P}^2}(1). \tag{29}$$

where

$$a = \begin{pmatrix} z_0 X - z_1 \\ z_0 Y - z_2 \\ z_0 J \end{pmatrix} \tag{30}$$

and

$$b = \begin{pmatrix} -z_0 Y + z_2 & z_0 Y - z_1 & z_0 I \end{pmatrix}. \tag{31}$$

Now to go from the sheaf theoretic description to the description in terms of matrices we simply restrict this sequence to $\mathbf{P}^2 \setminus \ell_\infty \simeq \mathbf{A}^2$. The following lemma completes the result.

Lemma 7.2.2. *Suppose that* (X, Y, I, J) *satisfy* $[X, Y] + IJ = 0$*. Then*

(1) $\ker a|_{\mathbf{A}^2} = 0$.
(2) $b|_{\mathbf{A}^2}$ *is surjective if and only if the stability condition holds.*

Bibliography

[Nak99] H. Nakajima. *Lectures on Hilbert schemes of points on surfaces*. Vol. 18. University Lecture Series. American Mathematical Society, Providence, RI, 1999, pp. xii+132. URL: https://doi.org/10.1090/ulect/018.