LECTURE 1

The Springer resolution

Today we will consider an example of a symplectic resolution of singularities. Recall that we are in the following situation. We have a Hamiltonian action of a reductive group G on a symplectic variety X. We can consider the following flavors of Hamiltonian reduction of the Hamiltonian G action on the cotangent bundle T^*X:

$$\mathcal{M}_0 = \mu^{-1}(0) /\!/ G \tag{1}$$

where $\mu\colon X \to \mathfrak{g}^*$ is the moment map and its twisted version

$$\mathcal{M}_\chi = \mu^{-1}(0) /\!/_\chi G \tag{2}$$

where $\chi\colon G \to \mathbf{C}^\times$ is a character. There is a canonical map

$$\pi\colon \mathcal{M}_\chi \to \mathcal{M}_0. \tag{3}$$

By the result we stated last time we see that when $\mathcal{M}_\chi$ is non-singular then this map is a resolution of singularities with the additional conditions that

- $\mathcal{M}_\chi$ is symplectic and
- $\mathcal{M}_0$ is Poisson and the map π is a Poisson map.

We call such a resolution of singularities a *symplectic resolution* of singularities.

1.1. A reminder of the A_1 case

Consider $X = \mathbf{A}^2$ with its standard $G = \mathbf{C}^\times$ action by scaling. Then $\mathbf{C}^\times$ acts on the symplectic variety

$$\mathrm{T}^*X = \{(i,j) \mid i \in (\mathbf{C}^2)^*, j \in \mathbf{C}^2\} \tag{4}$$

in a Hamiltonian way with moment map $\mu(i,j) = ij$.

We have seen that the affine GIT reduction of $\mu^{-1}(0)$ by $\mathbf{C}^\times$ is the following quadric

$$\mathcal{M}_0 \stackrel{\text{def}}{=} \mu^{-1}(0) /\!/ \mathbf{C}^\times \simeq Q = \{(a,b,c) \mid a^2 + bc = 0\} \tag{5}$$

Also, for $\chi(\lambda) = \lambda$ we have seen that the twisted GIT reduction is

$$\mathcal{M}_\chi = \mu^{-1}(0) /\!/_\chi \mathbf{C}^\times \simeq \mathrm{T}^*\mathbf{P}^1. \tag{6}$$

Furthermore, we have the resolution of singularities

$$\pi\colon \mathrm{T}^*\mathbf{P}^1 \to Q. \tag{7}$$

Of course, $T^*\mathbf{P}^1$ is naturally a symplectic manifold. There is also a Poisson structure on Q defined by

$$\{b,c\} = 2a, \quad \{a,b\} = b, \quad \{a,c\} = -c. \tag{8}$$

Each of these structures is compatible with the standard symplectic structure on $T^*\mathbf{A}^2$, and furthermore the map π is a Poisson morphism.

1.2. THE SPRINGER RESOLUTION

We consider a generalization of this example. In the remainder of this section we take $G = SL(n, \mathbf{C})$, but any semi-simple reductive group will work. Consider the ***nilpotent cone*** defined by

$$\mathcal{N} \stackrel{\text{def}}{=} \{a \in \mathfrak{g} \mid a^N = 0, \quad \text{for some } N\} \subset \mathfrak{g}. \tag{9}$$

Then $\mathcal{N}$ is an affine algebraic variety and it is equipped with a $\mathbf{C}^\times$ action $a \mapsto \lambda a$ (this is why it is called a 'cone'). When $G = SL(2, \mathbf{C})$ then we have $\mathcal{N} = Q$ from the previous example. In general, $\mathcal{N}$ is a singular affine variety with cone point $0 \in \mathcal{N}$.

Consider a maximal torus $T \subset G$ with Lie algebra $\mathfrak{h}$. Let $B \subset G$ be a Borel subgroup containing T. In the case $G = SL(n, \mathbf{C})$ we can take B to be the subgroup of upper triangular matrices. Define the ***flag variety*** of G to be

$$\mathcal{F} \stackrel{\text{def}}{=} G/B \tag{10}$$

It is isomorphic to the variety of all Borel subgroups of G. In the case $G = SL(n, \mathbf{C})$ this is isomorphic to the set of full flags of n-dimensional space

$$0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n \simeq \mathbf{C}^n \tag{11}$$

where $\dim V_i = i$. We will denote such a flag by $V_\bullet \in \mathcal{F}$.

Consider the special case of $G = SL(n, \mathbf{C})$. Let

$$\widetilde{\mathcal{N}} \stackrel{\text{def}}{=} \{(V_\bullet, y) \mid yV_i \subset V_{i-1}\} \subset \mathcal{F} \times \mathfrak{sl}(n, \mathbf{C}). \tag{12}$$

Notice that the condition $yV_i \subset V_{i-1}$ implies that y is nilpotent, thus

$$\widetilde{\mathcal{N}} \subset \mathcal{F} \times \mathcal{N}. \tag{13}$$

If we think about $\mathcal{F}$ instead as the space of all Borel subalgebras then $\widetilde{\mathcal{N}}$ is the set of pairs $(\mathfrak{b}, y)$ such that $y \in \mathfrak{b}$. The projection $\widetilde{\mathcal{N}} \to \mathcal{F}$ is a vector bundle with fiber $[\mathfrak{b}, \mathfrak{b}] = \mathfrak{n}$.

Proposition 1.2.1. *There is an isomorphism $\widetilde{\mathcal{N}} \simeq T^*\mathcal{F}$. In particular, $\widetilde{\mathcal{N}}$ carries a symplectic structure. Further, the natural left action of $SL(n, \mathbf{C})$ on $\widetilde{\mathcal{N}}$ is Hamiltonian with moment map*

$$\mu \colon \widetilde{\mathcal{N}} \to \mathfrak{sl}(n, \mathbf{C}) \tag{14}$$

defined by $\mu(V, y) = y$.

PROOF. Fix a basis $\{e_1, \ldots, e_n\}$ on $\mathbf{C}^n$ and let $V_\bullet^0$ denote the standard flag given by

$$V_i^0 = \text{span}\{e_1, \ldots, e_i\}. \tag{15}$$

Then the Borel subgroup B is

$$B = \{g \in SL(n, \mathbf{C}) \mid g \cdot V_i^0 \subset V_i\}. \tag{16}$$

Now, from example **??** we have an identification

$$\mathrm{T}^*\mathcal{F} = \{(V, y) \mid \mathrm{tr}(ay) = 0, \text{for any } a \text{ with } a \cdot V_i \subset V_i\} \subset \mathcal{F} \times \mathfrak{sl}(n, \mathbf{C}). \tag{17}$$

This implies that $\widetilde{\mathcal{N}} \simeq \mathrm{T}^*\mathcal{F}$ as desired. □

Since the image of the moment map μ is contained in the nilpotent elements we see that it defines a map

$$\mu\colon \widetilde{\mathcal{N}} \to \mathcal{N}. \tag{18}$$

It turns out that this is a symplectic resolution of singularities. For more details we refer to [CG10].

1.3. Springer resolution in general

We briefly sketch how this construction is generalized to the case of an arbitrary semisimple group G.

Lemma 1.3.1. *There is an isomorphism*

$$\mathrm{T}(\mathcal{F}) \simeq G \times^B \mathfrak{g}/\mathfrak{b} \tag{19}$$

Proof. Consider the trivial vector bundle on the G/B with fiber $\mathfrak{g}$. There is a surjective map of vector bundles

$$L\colon G/B \times \mathfrak{g} \to \mathrm{T}(G/B) \tag{20}$$

which sends a pair (gB, x) to the pair $(gB, \xi_x(gB))$ where ξ_x is the vector field on G/B determined by the infinitesimal left action of $x \in \mathfrak{g}$. The kernel of this map is the vector bundle whose fiber over gB is the Lie algebra of the stabilizer of gB; which is the Borel subalgebra $\mathrm{Ad}_g\mathfrak{b}$. There is an isomorphism

$$G \times^B \mathfrak{b} \to \ker L \tag{21}$$

sending $[g, x] \mapsto (gB, gxg^{-1})$. Thus we have an isomorphism of vector bundles

$$\mathrm{T}(G/B) \simeq (G/B \times \mathfrak{g})/(G \times^B \mathfrak{b}) \simeq G \times^B \mathfrak{g}/\mathfrak{b} \tag{22}$$

as desired. □

As a consequence we have

$$\mathrm{T}^*(G/B) \simeq G \times^B (\mathfrak{g}/\mathfrak{b})^*. \tag{23}$$

Since G is semisimple we identify this with

$$\mathrm{T}^*(G/B) \simeq G \times^B \mathfrak{b}^\perp, \tag{24}$$

where $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{b}^\perp$ is the orthogonal decomposition with respect to the Killing form. Furthermore, there is a direct sum decomposition $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ where $\mathfrak{n} \subset \mathfrak{b}$ is the subalgebra of nilpotent elements in $\mathfrak{b}$. One can show that $\mathfrak{n} \subset \mathfrak{b}^\perp$, so that $\mathrm{T}^*(G/B) \simeq G \times^B \mathfrak{n}$.

Now, G acts on G/B so it automatically acts on $\mathrm{T}^*(G/B)$ in a Hamiltonian way. Explicitly, the moment map

$$\mu\colon G \times^B \mathfrak{n} \to \mathfrak{g}^* \simeq \mathfrak{g} \tag{25}$$

is $\mu([g,x]) = gxg^{-1}$. Again, it is immediate to see that this map lands in the set of nilpotent elements $\mathcal{N} \subset \mathfrak{g}$, so it defines a map

$$\mu\colon \mathrm{T}^*(G/B) \to \mathcal{N}. \tag{26}$$

Again, this is a symplectic resolution of singularities.

1.4. Kähler quotients

We return to the relationship between the quotient of a vector space by a real compact Lie group and the GIT reduction by the corresponding complex reductive group. Let K be a compact real Lie group and let G be its complex form. We assume that $K \subset G$ and that the complexification $K(\mathbf{C}) \simeq G$. Notice that at the level of Lie algebras we have $\mathfrak{g} = \mathfrak{k} \otimes_{\mathbf{R}} \mathbf{C}$.

Let $V_{\mathbf{R}}$ be a real vector space equipped with a positive definite inner product g and a compatible complex structure I. When we chose to view $V_{\mathbf{R}}$ as a complex vector space via I we denote it by V. This equips $V_{\mathbf{R}}$ with the structure of a Kähler manifold. Let $(-,-)$ be the corresponding Hermitian inner product and let $\omega = \mathrm{Im}(-,-) \in \wedge^2 V_{\mathbf{R}}^*$ be the symplectic form. Suppose that $K \subset U(V_{\mathbf{R}})$ acts unitarily on V and hence its complexification $G \subset GL(V,\mathbf{C})$ acts through complex linear transformations.

By assumption, the action of K is symplectic with respect to ω. Thus we have a moment map

$$\mu_{\mathbf{R}}\colon V_{\mathbf{R}} \to \mathfrak{k}^* \tag{27}$$

defined explicitly by the rule that

$$\langle \mu_{\mathbf{R}}(v), a\rangle = \frac{1}{2}\omega(x, a\cdot x). \tag{28}$$

But, since ω is the imaginary part of the Hermitian inner product, we can write the moment map as $\langle \mu_{\mathbf{R}}(v), a\rangle = \frac{i}{2}(a\cdot x, x)$.

In this situation we can contemplate two quotients, namely $\mu_{\mathbf{R}}^{-1}(0)/K$ and $V /\!/ G$. (Recall that generally the G-orbit of an arbitrary element $v \in V$ will not be G-closed, so there is no expected relationship between V/K and $V /\!/ G$.) The Kempf–Ness theorem states that these quotients can naturally be identified.

Theorem 1.4.1 ([KN79]). *The G-orbit of any $x \in \mu_{\mathbf{R}}^{-1}(0)$ in V is closed and furthermore there is an isomorphism*

$$\mu_{\mathbf{R}}^{-1}(0)/K \xrightarrow{\simeq} V /\!/ G \tag{29}$$

intertwining the complex structures.

We will not prove this theorem, but let us unpack the last statement. By definition, the space $\mu^{-1}(0)/K$ is the real Hamiltonian reduction of V by the K-action. It inherits a Kähler structure from that on V. On the other hand, the right hand side $V /\!/ G$ is the affine GIT quotient, which is a complex affine variety by definition.

Bibliography

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