LECTURE 7

Symplectic reduction, I

We turn to an alternative description of affine (and twisted[1]) GIT quotients. Let's motivate this by considering a simple example.

7.1. Reductive versus unitary

Let V be a vector space equipped with a hermitian metric. Let $G \subset U(V)$ be a connected closed Lie group acting by unitary transformations on V.

Warning: In this section G denotes a real compact Lie group. We will denote its complex form by $G^{\mathbf{C}}$.

We have defined the affine GIT quotient

$$V /\!/ G^{\mathbf{C}} = \operatorname{Spec} \mathbf{C}[V]^{G^{\mathbf{C}}}. \tag{1}$$

The underlying space consists of the set of closed $G^{\mathbf{C}}$-orbits. Let's consider the example

$$V = \operatorname{End}(\mathbf{C}^n) \tag{2}$$

with $g \in G = U(n)$ acting by conjugation $g \cdot B = g^{-1}Bg$. In this case, $G^{\mathbf{C}} = GL(n, \mathbf{C})$, and we have identified the closed $GL(n, \mathbf{C})$-orbits: a matrix has a closed $G^{\mathbf{C}}$-orbit if and only if it is diagonalizable. Hence

$$V /\!/ G^{\mathbf{C}} = \operatorname{End}(\mathbf{C}^n) /\!/ GL(n, \mathbf{C}) \simeq \mathbf{C}^n. \tag{3}$$

On the other hand, a matrix B satisfies

$$[B, B^\dagger] = 0 \tag{4}$$

if and only if it can be diagonalized by a unitary matrix (to see this use Schur's lemma which states that any complex square matrix is unitary equivalent to an upper triangular matrix). Thus there is a bijection

$$\operatorname{End}(\mathbf{C}^n) /\!/ GL(n, \mathbf{C}) \simeq \{B \in \operatorname{End}(\mathbf{C}^n) \mid [B, B^\dagger] = 0\}/U(n). \tag{5}$$

If we let

$$\mu \colon V \to \mathfrak{g}^* \tag{6}$$

be $\mu(B) = \frac{i}{2}[B, B^\dagger]$ (where we use the hermitian form to identify $\mathfrak{g} = Lie(U(n))$ with $\mathfrak{g}^*$) then we can rewrite this as

$$V /\!/ G^{\mathbf{C}} \simeq \mu^{-1}(0)/U(n). \tag{7}$$

[1] We introduce twisted GIT quotients today

The right hand side is called the symplectic, or Hamiltonian, reduction We will see this is a general feature about affine GIT quotients. Before that, we introduce a slight variant of the affine GIT quotient.

7.2. Twisted GIT quotient

For an affine algebraic variety X which is acted on by a reductive group G we have seen that the GIT quotient $X /\!/ G$ is in bijective correspondence with the set of closed G-orbits.

A projective variety $X \subset \mathbf{P}^n$ has a canonical $\mathbf{Z}_{\geq 0}$-grading on its algebra of global functions

$$A_\bullet = \oplus_{n \geq 0} A_n \tag{8}$$

where A_n is the algebra of degree n homogenous polynomials restricted to X. One can recover X from the graded algebra $A_\bullet$ via the 'proj' construction

$$X = \mathrm{Proj}(X). \tag{9}$$

The closed points of $\mathrm{Proj}(X)$ correspond to the set of graded ideals $J_\bullet \subset A_\bullet$ which are maximal among graded ideals not containing $A_+ = \oplus_{n>0} A_n$.

Let us return back to the situation of a reductive group G acting on an affine algebraic variety X. Suppose that χ is a character of G, meaning a homomorphism $\chi\colon G \to \mathbf{C}^\times$. Define the space of χ-twisted invariant functions to be

$$\mathbf{C}[X]^{G,\chi} \overset{\text{def}}{=} \{f \in \mathbf{C}[X] \mid f(g \cdot x) = \chi(g) f(x)\} \subset \mathbf{C}[X]. \tag{10}$$

Notice that when $\chi = 1$ then this return the usual G-invariants, but in general $\mathbf{C}[X]^{G,\chi}$ is not an algebra.

Even though $\mathbf{C}[X]^{G,\chi}$ is not an algebra, we obtain a canonical graded algebra by the formula

$$A_\bullet = \oplus_{n \geq 0} \mathbf{C}[X]^{G,\chi^n}. \tag{11}$$

Notice that $A_0 = \mathbf{C}[X]^G$ is the usual algebra of invariants functions.

Definition 7.2.1. The ***twisted GIT quotient*** is the quasi-projective variety

$$M /\!/_\chi G \overset{\text{def}}{=} \mathrm{Proj}\left(\oplus_{n \geq 0} \mathbf{C}[X]^{G,\chi^n}\right) \tag{12}$$

The canonical map $\mathbf{C}[X]^G \to \oplus_{n \geq 0} \mathbf{C}[X]^{G,\chi^n}$ induces a projective map

$$\pi\colon X /\!/_\chi G \to X /\!/ G. \tag{13}$$

Consider a character χ as above. Extend the G-action on X to a G-action on the total space of the trivial line bundle $X \times \mathbf{C}$ by the formula

$$g \cdot (x, \mu) = (g \cdot x, \chi(g^{-1})\mu). \tag{14}$$

A point $x \in X$ is called ***χ-semistable*** if for any $\mu \in \mathbf{C}^\times \subset \mathbf{C}$ the closure of the orbit of (x, μ) is disjoint from the zero section:

$$\overline{\mathbb{O}}_{(x,\mu)} \cap (X \times \{0\}) = \emptyset. \tag{15}$$

An equivalent way to understand semistability is the following.

Lemma 7.2.2. *A point $x \in X$ is χ-semistable if and only if there exists an $f \in \mathbf{C}[X]^{G,\chi^n}$, for some $n \geq 1$, such that $f(x) \neq 0$.*

From this characterization it is easy to see that the set of χ-semistable elements

$$X_\chi^{ss} \subset X \tag{16}$$

is G-invariant.

THEOREM 7.2.3. *Suppose that G is a reductive group acting on an affine variety X and let χ be a character of G. Then*

1. *The map $X /\!/_\chi G \to X /\!/ G$ is surjective.*
2. *As topological spaces one has*

$$X /\!/_\chi G \simeq X_\chi^{ss} / \sim \tag{17}$$

where $\mathbb{O} \sim \mathbb{O}'$ iff $\overline{\mathbb{O}} \cap \overline{\mathbb{O}}' \cap X^{ss} \neq \emptyset$.

3. *There is a bijection*

$$X /\!/_\chi G \simeq \{\textit{closed orbits in } X^{ss}\}. \tag{18}$$

Notice that orbits which are closed in X^{ss} may not be closed in X, so that the twisted GIT reduction has the potential to see a wider class of orbits.

Example 7.2.4. Consider the $\mathbf{C}^\times$ action on $\mathbf{A}^2$ which scales each direction the same. Then we have seen that

$$\mathbf{A}^2/\mathbf{C}^\times = \{0\} \cup \mathbf{P}^1, \quad \mathbf{A}^2 /\!/ \mathbf{C}^\times = \{0\}. \tag{19}$$

Let $\chi(\lambda) = \lambda$. Then

$$(\mathbf{A}^2)_\chi^{ss} = \mathbf{A}^2 \setminus 0, \tag{20}$$

and hence

$$\mathbf{A}^2 /\!/_\chi \mathbf{C}^\times = \mathbf{P}^1. \tag{21}$$